2mix

NASA TECHNICAL TRANSLATION NASA TT F-15,404

THEORY OF OSCILLATION OF THE EARTH'S ATMOSPHERE

L. A. DIKIY

Translation of "Teoriya kolebaniy zemnoy atmosfery," Leningrad, Hydrometeorological Press, 1969, pp 1-195

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 FEBRUARY 1974

TT 71-53152
NASA TT F-15,404

Teoriya kolebaniy zemnoy atmosfery
Gidrometeorologicheskoye Press, Leningrad,
1969

Dikiy, L.A.

THEORY OF OSCILLATION OF THE EARTH'S ATMOSPHERE

Translated from Russian

Prepared for the National Aeronautics and Space Administration and National Science Foundation, Washington, D.C. by Mrs. Geti Saad.

UDK 551.511.3

Dikii L.A.

THEORY OF OSCILLATIONS OF THE EARTH'S ATMOSPHERE.

(Teoriya kolebanii zemoi atmosfery)

Gidrometeorologicheskoe izdatel 'stvo

Leningrad, 1969.

Translated from Russian.

C O N T E N T S

		Page No.
Introduction		1
Chapter-1	Equations of the Theory of Disturbances	11
2.	Energy Potential Vortex	15
3.	Laplace's equation of the theory of tides	19
4.	Three-dimensional case. Equation for divergence	22
5.	Division of variables	25
6.	Peckeris solution	29
Chapter-2	Laplacian Equation of the Theory of Tides	34
1.	Laplace's equation. Integral relations	34
2.	Fundamental curves. Asymptotics at low	38
3.	Conversion to type convenient for application of Galerkin's method	43
4.	Calculation of characteristic curves of Laplace's equation of the theory of tides	48
5.	Results of characteristic curves	59
6.	Wave type and the number of resolution points	67
7.	A note on expansion according to Hough's functions	71

Chapter-3 Characteristic Asymptotic Curves of Laplace's Equation of the Theory of Tides for low Dynamically Equivalent Depths. 84

1. Working System of Equations 84

2. The approximating Resolutions 88

3. Conjugation of Asymptotic Formulas in a Complex Plane 95

4. Two Branches of Asymptotes 99

5. Correctness of carried out Approximation. Additional Wave, Propagating Eastward 104

6. Negative Values of Equivalent Depth 110

7. Conjugation of Asymptotes at High and Low 123

Chapter-4 Vertical Structure of Oscillations 128

1. The case of isothermal stratification 128

2. Acoustic and Gravitational waves 133

3. Gyroscopic-inertia waves 137

4. The case of real stratification 142

5. Calculation of characteristic curves 145

6. Short-wave asymptotes 152

7. Long waves 156

8. Velocities of Rossbi waves. Comarison with empirical data 165

Chapter-5 Energy of Oscillations 168

1. Energy and classification of waves 168

2. Theorem of virial 173

3. Energy composition of oscillations 184

4. Results of numerical calculations. Atmospheric wave guides. 189

Chapter-6 Expansion according Natural Oscillations 202

1. Two-fold Completeness of Eigen Functions 202

2. Proof of Completeness. Boundary Condition Y = O. 209

3. Auxiliary Formulas and Asymptotic Evaluations 215

4. The proof of Completeness. The Real Marginal Condition 220

5. Expansion Formulas 231

Chapter-7 Propagation of Disturbance from Instantaneous Point Source 236

1. Propagation Velocity of Disturbance 236

2. Function of Koshi Problem Effect 240

3. Acoustical and Gravitational Parts of Resolutions. 248

4. Asymptotic Behavior of Effect Function at High t Values 253

5. Numerical Estimate Results of the Weighting Function 263

6. Formulation of Problem on Propagation of Disturbances in the case of actual Temperature Profile and Spherical Earth 267

Chapter-8 Accounting for the Medium Wind in the Problem of Frequencies on the Atmosphere's Free Oscillations. 279

-----oOo-----

INTRODUCTION.

Investigation of a system of equations, depicting dynamics of atmosphere, is one of the most complex problems of hydromechanics. Combination of movements different in character and scale, the enormous role of hydrodynamic processes and energy conversion make it impossible to use majority of simplifications usual for other problems. Atmosphere as an actual physical system, to which it is possible to apply precise quantitative methods, was first systematically investigated in the work of M. Margules and specially in the work of the Norwegian school scientists. Many results of this period retain their significance even at present; their systematic account could be found in the book of Bjerknes V., Solberg H., Bergeron T., 1933. Great contribution to the study of the dynamics of atmosphere was made By Soviet Scientists A.A. Fridman and N.E. Kochin.

At present almost the only effective method for studying the complete system of equations of atmosperic movements remains numerical integration, and the problem does not yield to any complete and exact logical analysis. Many significant features of atmospheric movements could be defined in a simple linear model. The atmosphere could be analyzed as a fine film on revolving sphere having definite elastic properties. Within this film originate the waves, which sometimes envelope the whole atmosphere. If the phase velocity of wave is very much higher than the velocity of particles within the wave, linear approximation gives very accurate result; otherwise only very approximate features of the dynamics of atmosphere could be defined.

Atmosphere, taken as a film, is a very complex oscillating system. Its elasticity is epecified by many causes. Besides the fact, that even with most idealized assumptions, this medium is not of simple structure, air particles are affected by forces of various nature. Firstly, it is forces of ordinary elasticity, bound with compressibility of air. Secondly these are the buoyancy forces, specified by non-uniformity of atmosphere in elevation, layering or stratification. Particle, deflected vertically, if its state varies adiabatically, acquires density different from surrounding particles. The difference Archmedes force and its weight forces it either to continue deviation from original state with acceleration, or to return into initial state, due to which there are oscillations close to equilibrium. In the first case it is known as unstable stratification, in the second - stable.

For stability it is required and is sufficient, that the temperature drop with altitude in atmosphere should occur gradually, slower than the temperature drop in a particle adiabatically displaced upward. On an average the atmosphere is always stable, there could only be individual zones of instability, mainly close to the surface of Earth, where the convection currents are being developed. The monograph will analyse a certain averaged model of atmosphere, and therefore, it will always be stable. Thirdly, the atmosphere has some gyroscopic rigidity, reacting to any disturbance by the appearance in it of oscillating motion, as in precessing top, since the atmosphere rotates jointly with Earth, representing its own type of gyroscope.

Gyroscopic forces cannot be felt with motion on a very small scale; in the study of these movements it may be assumed, that the atmosphere is on an average immobile and that the Earth is flat. Here it is possible to separate waves with periods with periods 5-10 min., bound mainly with the effect of gravity forces, i.e. of buoyancy force. For these waves, which are called the short gravity waves, elasticity is a non-essential factor, its influence is low. For movements on very large scale (such, as cyclonic vortex) buoyancy force cannot be of any significance. Here the determining factors are the gyroscopic force, although the elasticity also has some effect.

Thus, the different physical nature of these forces results in movements, absolutely different in structure and scale, which correspond to them. This state is found to be convenient. It permits to study each motion independently, i.e. in the study, for instance, of sound waves to disregard both the Earth's rotation and gravity, and in the study of short gravity waves-compressibility. Appropriate simplifications are also being done by meteorologists, in the investigation of large-scale movements. This results every time in very negligible distortion of the type of waves being investigated and in considerable simplification of the system of equations, related with time reduction of its order. The hydrodynamic system of equations, initially of fifth order, becomes divided into two systems of second order, depicting acoustic and gravity waves respectively, and equation of first order for gyroscopic waves (inertia waves).

This is the procedure in all the cases, when investigation is required for one particular type of waves in the resolution of concrete applied problems. Meteorologist is interested primarily in the largest inertial-gyroscopic waves directly related with forecast of meteorological fields, forecast of large-scale weather background. He is also interested in gravity waves in the investigation of local events, The largest gravity waves are of primary importance in the investigation of atmospheric tides. Large-scale (synoptic scale) waves were studied extensively by E.N. Blinova and other investigators. Ample literature is also available on gravity waves, long and short, and questions of atmospheric acoustics, for instance, on propagation of waves from high intensity local disturbances. But from the point of view of the principle it would also be of interest without going into individual structural details of spacific oscillations, to review the spectre as a whole, defining the interdependence of its individual parts. In this case it is found, as should have been expected, that between the different types of waves there is no existence of very shape boundary. There are transient oscillations, which are affected simultaneously by several factors.

We are trying to give as complete as possible spectrum pattern of a system of equations of the fifth order. The problem of dividing the spectrum into individual parts-acoustic and gravity has been studied for quite sometime, starting from the above - mentioned monograph of V.Bjerkness and others (see also Eliassen, Kleinschmidt, 1957). In a more complete form it is given in an article by A.S. Monin and A.M. Obukhov (1958), and also in Eskart's monograph (1960). These works discussed the model of isothermal atmosphere above a flat Earth. In other works of the author (1961, 1965) this problem was generalized for the case of rotating spherical atmosphere with stratification, approximating the real one.

In the resolution of our problem the variables are divided, and we obtain a separate problem on eigenvalues for the equation, which includes only vertical coordinate, and also for the equation with horizontal coordinates. The second equation is quite independent of the model adopted for atmosphere's stratification and also remains the same for a composite atmosphere with variable temperature, for isothermal atmosphere and for a uniform ocean. This is the so called Laplace's equation (tidal equation), frequently applied in various problems of the sea and ocean physics. In spite of the " classicality " of this equation, its theory cannot be called completed. The main contribution to the theory of this equation was made by the English astronomer Hough, 1897, 1898. He has found the asymptotic solutions and suggested numerical method of solution for the case of an ocean of great depth. Hough's solutions were successfully applied to the problems of the dynamics of atmosphere's by Rossbi[1], Haurwitz, 1937, 1940, and by Blinova (1943). Blinova has shown, that asymptotic solutions of Laplace's equation in the theory of tides could also be successfully applied to explain the centres of the atmosphere's activity also for weather forecast. The corresponding solutions of Laplace's equation are denoted as Rossbi waves. The same asymptotes are, generally, sufficient to study the semi-diurnal oscillations, connected with the tides in the atmosphere (the reason, why the equation is called tidal).

However, if the need is to investigate the diurnal oscillations

---

[1] Rossbi has found his solutions independently, apparently, not knowing, that this is the ultimate case of Hough's resolutions for an ocean of of great depth.

of the atmosphere or any other problem connected with inner waves in atmosphere, or Rossbi waves in atmosphere not of the Earth, but of fast rotating big planet, for instance, of jupiter, or the required asymptotics of the Laplacian tidal equation in opposite extreme case, for low values of the equivalent ocean depth (this concept will be explained in Para 5 of chapter 1). This type of asymptote could be obtained and it will be found, that both the asymptotes jointly cover practically the whole possible range of values for the equivalent ocean depth.

Chapter 1 is introductory. The analysis in it is of the basic equations of the problem and the above mentioned division of variables is carried out.

Chapter 2 and 3 are devoted to the theory of Laplacian tidal equation, It should be mentioned, that here remains a lot which is not quite clear, for instance, the question regarding invariance of the number of nodes in fundamental function along the wave mode. True, even every incomplete results, obtained in chapter 3, permit to resolve theorerically the question regarding conjunction of the branches of the asymptotes at low and high values of equivalent depth. Actually, at present there is no real theory of special functions, connected with the tidal Laplac's equation of (Hough's functions), such a theory as available for generally-known special functions, with algebraic, integral and other relations between them.

Chapter 4 discusses equation, pertaining to vertical coordinate. The first paras present well known results in connection with isothermal atmosphere. In the following paragraphs a lot of attention is being paid to the main problems regarding disposition of eigen-curves on the plane of parameters for a more composite model of an atmosphere stratified in a real way. Since in this case we have an equation with variable factors, not admitting of a closed analytical resolution, the qualitative investigations and computer calculations most significant become. The deduction seems significant regarding alternation of eigen-curves of our marginal problem with a composite extreme condition, depending on self parameter, and of a more simple problem with a boundary condition independent of it. This result will be found indispensable in chapter 6 to prove the completeness of a system of eigen-functions.

In the investigation of an ultimate case of long waves instead of natural oscillations it is necessary to investigate the forced oscillations. This is carried out in the same way, as in Wilkes work (1949), but without the approximate quasistatistics, as done in that work. Because of this we can also investigate the region of higher frequencies and analyse the ultimate transition to quasistatistics. The correlating curve, obtained in chapter 4, contains the whole frequency spectrum, and unifies various spheres of investigation.

Division of waves into types could be carried out in different ways, It is possible, as it was done, for instance, by Monin and Obukhov, to follow the behavior of resolutions with variation of the atmosphere's parameters-compressibility and static stability factor - and in relation to this behavior to assign the resolutions to one or another (see chapter 4). This division could be done on the basis of various types of energies-kinetic, potential " thermobaric " and energy, bound with air elasticity (elastic energy)- in general the energy blance of oscillation. This is being dealt with in chapter 5, in which relations are obtained permitting without calculation to evaluate the share of various types of energy from the position of point, depicting the given oscillation on dispersion curve.

Energy composition of oscillation is not only a parameter, which permits to carry out in the most physical natural way the classification of oscillations; energy is included in many important formulas, for instance, it is bound by a simple relation with group velocity. By means of energy it is possible to clarify purely mathematical facts with regard to monotony of natural curves. It is the most suitable metrics for analytically-functional study of equations (chapter 6), whence ensues the proof of completeness and formulas for expansion according to fundamental functions. The same metrics is used for the formation of the perturbation theory for estimating perturbation of spectrum, caused by an average wind velocity, averaged in altitude; and the weight in this averaging is nothing else, but the energy density of undisturbed oscillation.

Thus, chapter 5 is devoted to an all round investigation of this most significant characteristic - energy. For instance, demonstration of the " theorem of virial " regarding equality of kinetic and potential energy average values in the absence of Earth's rotation is given and evaluation of the increasing share of kinetic energy with the Earth's rotation is made. The end of the chapter deals with the problem of atmospheric wave guides. Short waves are concentrated in separate layers of atmosphere : short acoustic waves in layers with the lowest adiabatic sound velocity, i.e. in cold layers at altitudes 17 and 84 km, whereas the short gravity waves in lower layers of the highest relative stability at altitudes 30 and 110 km.

It is interesting, that for long waves a unique inversion is evident- cold layers serve as energetic barriers. The pattern of the atmospheric wave guides given here is similar to that shown by Pfeffer and Zarichny (1963) and Press and Harkrider (1962), who obtained it for a narrower class of waves and frequency interval.

Chapter 6 according to the applied mathematical apparatus is somewhat different from the others. Here we demonstrate the completeness (two-fold) of the system of fundamental functions, obtained in chapter 6. This is a problem of " non-classical " type. The natural parameter is included in the equation in a split-linear way; it is also included into boundary condition. In this chapter the methods are of functional analysis.

However, reader, not interested in the purely mathematical side of the matter, could miss this chapter, taking on empirical formulas for expansion factors of fundamental functions, used in para 5 of chapter 7.

As pointed out in chapter 5, waves of various types have considerably different phase and group velocities of propagation. If in a small area of space at a certain moment there is a disturbance, from which waves disperse to all sides, a part of energy, bound with excitation of acoustic waves, will spread out very quickly and in a little while only the attenuating gravitational oscillations will remain in the part of space close to the source of disturbance.

In chapter 7 we find the asymptotes of waves, generating from an instant point source, confirming the just described intuitively obvious pattern. Asymptotic solutions are very similar to precise solutions given in the same chapter. Chapter 7 also shows how to apply the theory of expansion, in the fundamental functions evloved in preceding chapter, to the problem of disturbance propagation in spherical and realistically stratified atmosphere.

The content of chapter 8 has already been mentioned.

In conclusion we mention just a few words regarding the specifics of the present book. It is possible to investigate a very wide range of events on basis of appraising and approximating considerations, using mathematical apparatus as simple as possible. This may help obtain results useful for practical application. However, there are the works of others,more mathematical nature. We speak not only and not so much of more exact methods of solving the equations. Actually, the parameters of these equations are naturally known to be of not very high accuracy, and the equations themselves are sometimes over schematical. At a certain stage in this case the need emerges to bring up the theory to a certain degree of logical and mathematical precision and completeness with inevitable limitation of physical complexity of the problem. The present work is exactly in this direction, which obliged us to bring up the reasoning to some sensible degree of mathematical precision.

Thus, the development of mathematical formalism of theory bothered us more frequently, than the diversity of concrete geophysical applications. In this respect very similar to the existing works is a well known Eskart's book (1960) and an article by I. Tolstoy (1963), the content of which is only slightly resembling the contents of the present monograph.

Chapter. 1.

EQUATIONS OF THE THEORY OF DISTURBANCES.

1. Equation of motion.

We shall assume, that the atmosphere deviates a little from a certain mean state, which is a state of relative calm. In this state there are no velocities, and temperature, $\bar{T}$, pressure $\bar{p}$ and density $\bar{\rho}$ depend only on one coordinate - z the altitude. In this case they are connected by ratios

$$\bar{p} = \bar{\rho}\ \bar{R}\bar{T}, \quad \frac{\partial \bar{p}}{\partial z} = - g\bar{\rho},$$

the first of which is the equation of state, and the second - condition of static equilibrium. Thus, only one quantity, for instance, temperature $\bar{T}(z)$ remains free. Hence we shall be taking into consideration most frequantly the temperature of the so called standard atmosphere (Fig. 1.1.) CIRA 1961 (Cospar[1] International Reference Atmosphere). The figure shows, that the temperature curve has two depression - in stratosphere at an altitude of approximately 17 km and a deeper one in mesosphere at an altitude of about 84 km. At higher begins the continuous rise of atmosphere - thermosphere. We assume, that the atmosphere extends upto infinity and that the temperature rise is infinite. However, it should always be kept in view, that conclusions pertaining to

[1] Cospar - Committee on Space Research.

the bottom portion of atomsphere could have physical significance only, when they are independent or almost independent of the behavior of temperature curve at very high altitude, about 150 km and higher, where the applied equations represent extremely inaccurately the real movements of atmosphere. At these altitudes the significance is acquired by such factors as non-linearity of equations and viscosity.

Since the equations of motion are written in a spherical rotatory system of coordinates, there are additional terms - Coriolis acceleration. The Earth is assumed to be a smooth sphere of radius a, gravitational acceleration (which includes also centripetal moving acceleration) being constant and directed toward the centre of the Earth. In equations the quantities, which are the products of low deviation of the required fields from their average values are neglected, i.e. the equations are linearized. The obtained system of equations for disturbances of the first order has the following appearance :

$$\frac{\partial u}{\partial t} = - \frac{1}{\sigma \rho \sin\theta} \frac{\partial p}{\partial \varphi} - 2\omega\cos\theta\, v - 2\omega\sin\theta\, w, \quad (1.1)$$

$$\frac{\partial v}{\partial t} = - \frac{1}{\sigma \rho} \frac{\partial p}{\partial \theta} + 2\omega\cos\theta\, u, \quad (1.2)$$

$$\frac{\partial w}{\partial t} = - \frac{1}{p^{-}} \frac{\partial p}{\partial z} - g\frac{\rho^{'}}{\rho^{-}} - 2\omega\sin\theta\, v. \quad (1.3)$$

Here u,v,w - velocity components of the wind, directed respectively west to east, north to south and vertically upward; φ - longitude ; θ - latitude addition upto Ω /2; p', ρ' - deviations of pressure and density from their average values $\bar{p}$ and $\bar{\rho}$.

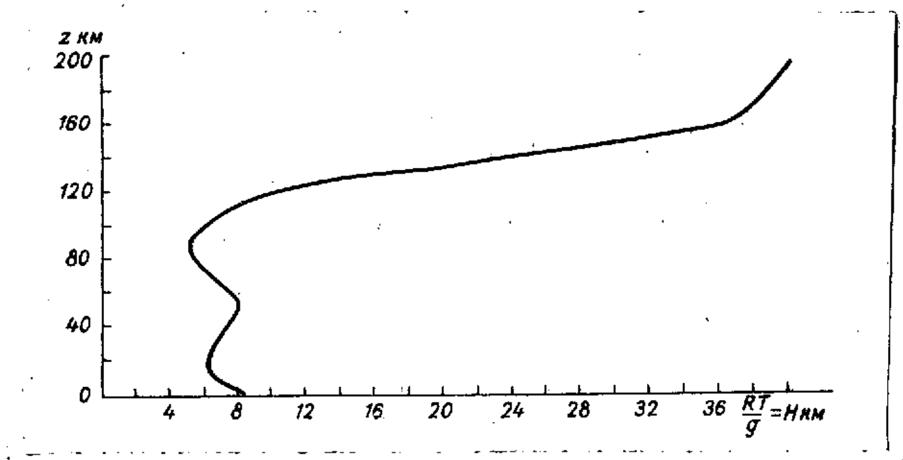

FIGURE : 1.1. Standard atmosphere CIRA 1961.

In meteorology it is the custom to neglect the Coriolis terms containing sin Θ. Eckart calls this neglecting as " traditional ". For the sake of simplicity we make simplification, which would allow to divide the variables. Apparently, this simplification does not affect very much the results, which is confirmed by the following reasoning. In equation (1.3) the Coriolis term is many orders lower than the gravitational acceleration g, therefore, its disregarding is quite justified.

In the first two equations the Coriolis terms could be of significance only in the study of the largest scale movements, And these movements are highly anisotropic. The horizontal scales of the fine film - earth atmosphere - exceed many times the vertical scales, correspondingly horizontal velocities are many times greater than the vertical one. This makes it possible to neglect the Coriolis terms, connected with the vertical velocities. These terms cannot neglected only in a very narrow belt around the equator, where cos Θ 0. It should be mentioned, that it would have been highly desirable to appraise more exactly the results of this traditional approximation, and the extent to which the perturbation equations, obtained without the Coriolis terms could be obtained by perturbation theory, by these terms.

To Euler's equations (1.1)-(1.3) it is also necessary to combine the equation of continuity

$$\frac{\partial \bar{\rho}}{\partial t} + w \frac{d\bar{\rho}}{dz} + \rho \bar{x} = 0. \tag{1.4}$$

Here X is taken to be the three - dimensional divergence

$$X = \frac{1}{a \sin\theta} \frac{\partial u}{\partial \varphi} + \frac{1}{a \sin\theta} \frac{\partial}{\partial \theta} (v \sin\theta) + \frac{\partial w}{\partial z}.$$

Finally, in order to close the system of equations, we require one more equation. We take for this the adiabatic condition, i.e. preservation of anisotropy

$$\frac{dp}{dt} - \frac{\varkappa p}{\rho} \frac{d\rho}{dt} = 0.$$

Linearizing this equation we get

$$\frac{\partial p'}{\partial t} + w \frac{d\bar{p}}{dz} - \frac{\varkappa \bar{p}}{\rho^-} \left(\frac{\partial p'}{\partial t} + w \frac{d\bar{\rho}}{dz} \right) = 0.$$

We denote $C^2 = \chi \bar{p}/\bar{p}$(C - adiabatic velocity of sound). Using equation of statics $d\bar{p} / dz = - g\bar{\rho}$ and substituting $\partial P'/\partial t$ from preceding equation, we get

$$\frac{\partial p'}{\partial t} = - C^2 px + g\omega \bar{p}. \tag{1.5}$$

Hence the dashes over p and $\bar{\rho}$ will be omited. Thus, we have the following system :

$$\frac{\partial u}{\partial t} = - \frac{1}{a\rho \sin} \frac{\partial p}{\partial \varphi} - 2\omega \cos\theta \vartheta; \qquad (1.1')$$

$$\frac{\partial v}{\partial t} = - \frac{1}{ap} \frac{\partial p}{\partial \theta} + 2\omega \cos \theta u; \qquad (1.2')$$

$$\frac{\partial w}{\partial t} = - \frac{1}{p} \frac{\partial p}{\partial z} - \frac{g\rho}{p}; \qquad (1.3')$$

$$\frac{\partial p}{\partial t} = - \bar{\rho} X - \frac{d\bar{\rho}}{dz} w; \qquad (1.4')$$

$$\frac{\partial p}{\partial t} = - c^2 \bar{p} X + g \bar{\rho} w. \qquad (1.5')$$

2. Energy. Potential vortex.

Now it is not difficult to write the law of conservation of energy for the system of equations (1.1')-(1.5'). For the sake of convenience we shall analyse complex solutions, keeping in view, that physical solutions will be obtained, if we take the imaginary portion of the complex **solutions**. It is easy to check by simple differentiation and substitution of time derivatives from equations of motion, that

$$\frac{\partial}{\partial t} \left\{ \bar{\rho} \frac{|u|^2 + |v|^2 + |w|^2}{2} + \frac{1}{2xp} \left[|p|^2 + \frac{g}{\beta} |p - c^2 p|^2 \right] \right\} =$$

$$= - \mathrm{Re} \left[\frac{1}{a \sin\theta} \frac{\partial}{\partial \varphi} (pu^*) + \frac{1}{a \sin\theta} \frac{\partial}{\partial \theta} (pv^* \sin\theta) + \frac{\partial}{\partial z} (pw^*) \right].$$

(1.6)

Here Re means, that imaginary part was taken, asterisk - complex conjugate.

$$\beta = (x - 1) g + \frac{dc^2}{dz} = xR \, (\gamma_a - \gamma), \qquad (1.7)$$

where $\gamma = - d\bar{T} / dz$ - temperature gradient, $\gamma_a = (x - 1)g/xR$ - adiabatic, or equilibrium temperature gradient, i.e. a gradient, at which particle, displaced adiabatically in the vertical direction has continuously the same temperature, as surrounding particles. Factor β is of high significance throughout the theory. It is denoted as the coefficient of static stability. By integrating (1.6) over a fixed volume the integral on the right-hand side gets transformed into a surface integral. If the integration of (1.6) is done over the whole space with appropriate boundary conditions, the expressions on the right and left hand side will be zero. This means, that the quantity

$$E = \int_0^{\infty} \int_0^{2\pi} \int_0^{\pi} \left\{ \bar{p} \, \frac{|u|^2 + |v|^2 + |w|^2}{2} + \frac{1}{2x\bar{p}} \, |p|^2 + \right.$$
$$\left. + \frac{g}{2x\bar{p}\beta} \left| p - c^2 p \right|^2 \right\} \sin\theta \, d\theta \, d\varphi \, dz \qquad (1.8)$$

is conserved. This is nothing else but energy, and therefore, in equation (1.6) on the left-hand portion - time derivative gives the energy density, and on the right - divergence of the vector gives the energy flux. In the formula for energy we shall distinguish four parts : kinetic energy of the horizontal component of motion

$$E_r = \int_0^{\infty} \int_0^{2\pi} \int_0^{\pi} \bar{p} \, \frac{|u|^2 + |v|^2}{2} \, \sin\theta \, d\theta \, d\varphi \, dz, \qquad (1.9)$$

kinetic energy of vertical component

$$E_B = \int_0 \int_0^{2\pi} \int_0^{\pi} \bar{p} \frac{|w|^2}{2} \sin\theta \, d\theta \, d\varphi \, dz \qquad (1.10)$$

(hence these two types of energy for brevity will be called horizontal and vertical energy), elastic energy, connected with pressure fluctuations,

$$E_Y = \int_0^{\infty} \int_0^{2\pi} \int_0^{\pi} \frac{1}{2x\bar{p}} |p|^2 \sin\theta \, d\theta \, d\varphi \, dz \qquad (1.11)$$

and, finally, energy connected with variations of entropy (thermobaric energy according Ekcart terminology),

$$E_T = \int_0^{\infty} \int_0^{2\pi} \int_0^{\pi} \frac{g}{2x\bar{p}\beta} |p - c^2 p|^2 \sin\theta \, d\theta \, d\varphi \, dz. \qquad (1.12)$$

The thermobaric energy is directly connected with the buoyancy forces, affecting particle which has deviated vertically from the state of equilibrium. If we assume the state of static equilibrium ($\beta = 0$) as basic and at the initial instant of movement the entropy fluctuation $\frac{C_v}{\bar{p}} |p - c^2\rho|$ equal to zero, then, firstly, they will remain equal to zero throughout the movement time, since from equations (1.4'), (1.5') it follows

$$\frac{\partial(p - c^2 p)}{\partial t} = -\beta \bar{\rho} w ,$$

and, secondly, thermobaric energy will be equal to zero, i.e. the sum of kinetic and elastic energy is conserved. This is the case of the so called autobarotropic flow.

Energy is a quadratic quantity in relation to variables. It is known, that, besides the quadratic conservable quantity, hydrodynamic equations of stratified fluid also admit a linear invariant - potential vortex, brought in more fully by Ertel (1942). For non-linear equations the potential vortex is equal to

$$\Omega = \frac{\text{grad } S \ (\text{rot } V + 2\omega)}{\rho},$$

where S - entropy, $S = c_v \ \text{ln} \ (p\rho^{-\chi})$. This quantity could again be multiplied by an arbitrary function of entropy f (S). Linearizing the equation of potential vortex conservation, $d\Omega / dt = 0$, we get for the linear system the equation of potential vortex of the following type

$$\frac{\partial \Omega_1}{\partial t} + v \frac{\partial \Omega_0}{a \partial \theta} + w \frac{\partial \Omega_0}{\partial z} = 0, \qquad (1.13)$$

where Ω_0 - potential vortex component of zero order in relation to disturbances, i.e. it is composed exclusively of values, pertaining to the basic stationary state, and Ω_1 - component of the first order. It is easy to check by direct differentiation and substitution of derivatives from equations (1.1')-(1.5'), that

$$\frac{\partial}{\partial t} \left\{ \left[\frac{1}{\sin\theta} \frac{\partial (\sin\theta \, u)}{a \partial \theta} - \frac{1}{\sin\theta} \frac{\partial v}{a \partial \varphi} \right] - \frac{1}{\bar{\rho}} \rho \right. +$$

$$+ \frac{1}{\bar{\rho}} \frac{\partial}{\partial z} \left(\frac{p - c^2 \rho}{\beta} \right) \Big] = v \frac{\partial l}{a \partial \theta}; \; l = 2\omega \cos\theta \, . \qquad (1.14)$$

This is the vortex equation. If the model of flat Earth is analyzed on the assumption of Coriolis acceleration invariability, we get the conservation law of the first order potential vortex,

$$\frac{\partial \Omega_1}{\partial t} = 0$$

an equation obtained by Monin and Obukhov (1958).

3. Laplace's equation of the theory of tides.

Let us take first the case of two-dimensional motion. For us now it will be just a formal model, more simple than the general case of three-dimensional motion. But to this model it is also possible to give a physical meaning, as it is usually done in meteorology, if it is taken into account, that the Earth atmosphere is a relatively fine film. Therefore, in the first approximation it could be assumed to be two-dimensional, averaging its parameters in thickness. Exactly the same equations are obtained in the study of surface waves in a uniform ocean. Two-dimensional equations could be formally obtained from three - dimensional, assuming vertical velocity to be equal to zero, and the other parameters independent of altitude. Equations (1.1'),(1.2') and (1.5') give

$$\frac{\partial u}{\partial t} = - \frac{1}{a\bar{\rho} \sin\theta} \frac{\partial p}{\partial \varphi} - 2\omega \cos\theta \, v,$$

$$\frac{\partial v}{\partial t} = - \frac{1}{a\bar{\rho}} \frac{\partial p}{\partial \theta} + 2\omega \cos\theta \, u,$$

$$\frac{\partial p}{\partial t} = - c^2 \bar{p} X. \qquad (1.15)$$

To start with for the sake of simplicity the Earth's rotation is not taken into account. Then in this system it will be necessary to assume, that $\omega = 0$:

$$U_t = - \frac{1}{a\bar{p}\sin\theta}\frac{\partial p}{\partial \varphi},$$

$$v_t = - \frac{1}{a\bar{\rho}}\frac{\partial p}{\partial \theta},$$

$$P_t = - C^2\bar{p}X. \tag{1.16}$$

By differentiating the third equation by t and excluding u and v by means of the first two equations, we get the wave equation

$$Ptt = C^2 \frac{1}{a^2}\left\{\frac{1}{\sin^2\theta}\frac{\partial^2 p}{\partial \varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial p}{\partial\theta}\right\}\left(= \frac{c^2}{a^2}\Delta p\right).$$

If the resolutions are assumed to be dependent on t by exponential law $e^{i\ \ t}$, the p will meet the equation

$$-\sigma^2\rho = \frac{c^2}{a^2}\Delta p. \tag{1.17}$$

In other words, p should be the fundamental function of Laplacian operator. Obviously, the Laplacian operator did not result by chance. For system (1.16) there is separate direction, it is invariant in respect of selecting spherical coordinates, or invariant in relation to swinging of sphere. Therefore, differential equation of the second order for the scalar value should also be invariant in relation to these swings. The most common equation of this type is the equation (1.17).

Thus, the appearance of equation (1.17) has a purely algebraic cause.

A different pattern is evident for the system of equations (1.15). Here there is one separated direction - direction of the axis of rotation of the Earth's. The system is no longer invariant in relation to any swings, but variant only in relation to swings around this axis. We carry out the same procedure of excluding u and v, in the case where it is done at once in the simplest way, assuming dependence of resolutions on t exponential. From the first two equations

$$i\sigma u = -\frac{1}{a\bar{\rho}\sin\theta}\frac{\partial p}{\partial\varphi} - 2\omega\cos\theta\, v,$$

$$i\sigma v = -\frac{1}{a\bar{\rho}}\frac{\partial p}{\partial\theta} + 2\omega\cos\theta\, u$$

we express algebraically u and v through p :

$$u = \frac{i\sigma}{4a\omega^2(f^2-\cos^2\theta)}\left[i\,\frac{\cos\theta}{f}\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right]\frac{p}{\bar{p}}\,.$$

$$v = \frac{\sigma}{4a\omega^2(f^2-\cos^2\theta)}\left[i\,\frac{\partial}{\partial\theta} + \frac{\operatorname{ctg}\theta}{f}\frac{\partial}{\partial\varphi}\right]\frac{p}{\bar{p}}\,. \qquad (1.18)$$

Here $f = \sigma / 2\omega$ - dimensionless frequency. It is inverse to period of oscillations, expressed as semidiurnal.

Substituting u and v into third equation we get

$$-\sigma^2 p = C^2\, FP, \qquad (1.19)$$

where

$$F = \frac{f^2}{\sin\theta}\,\frac{\partial}{\partial\theta}\left[\frac{\sin\theta}{f^2-\cos^2\theta}\left(\frac{\partial}{\partial\theta} \quad \frac{i\ ctg\theta}{f}\,\frac{\partial}{\partial\varphi}\right)\right] +$$

$$+\ \frac{f^2}{f^2-\cos^2\theta}\left[\frac{i\ ctg\theta}{f}\,\frac{\partial^2}{\partial\theta\,\partial\varphi} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\varphi^2}\right]. \tag{1.20}$$

Operator F could also be written in the following way :

$$F = \frac{f^2}{f^2-\cos^2}\,\Delta - \frac{f^2\sin 2\theta}{(f^2-\cos^2\theta)^2}\,\frac{\partial}{\partial\theta} + \frac{if\ (f^2+\cos^2\theta)}{(f^2-\cos^2\theta)^2}\,\frac{\partial}{\partial\varphi},$$

where Δ - the usual Laplacian operator on single sphere. Hence it is clear, that operator F- is Laplacian Δ, perturbed by terms, which convert into zero, when $\omega = 0$ (i.e. $f = \infty$). This operator generalizes Laplacian in anisotropic case. Equation (1.19) bears the name of Laplace equation of the theory of tides, or tidal Laplace equation, and the operator F- tidal Laplacian.

4. Three - dimensional case. Equation for divergence.

Let us go back now to the general case of three-dimensional equations. For solutions, depending on t according to the law $e^{i\sigma t}$, we have a system of equations :

$$i\sigma u = -\frac{1}{a\bar{\rho}\sin\theta}\,\frac{\partial}{\partial}\ 2\omega\cos\theta\, v, \tag{1.21}$$

$$i\sigma v = -\frac{1}{a\bar{\rho}}\,\frac{\partial p}{\partial\theta} + 2\omega\cos\theta\, u, \tag{1.22}$$

$$i\sigma w = -\frac{1}{\bar{\rho}}\,\frac{\partial p}{\partial z} - g\,\frac{\rho}{\bar{\rho}}, \tag{1.23}$$

$$i\sigma\rho = -\bar{\rho}X - \frac{d\bar{\rho}}{dz}W, \tag{1.24}$$

$$i\sigma p = -c^2\bar{\rho}X + g\bar{\rho}W. \tag{1.25}$$

Here σ could only be material, since the energy has to be preserved and there cannot be indefinitely increasing or transient solutions.

We solve the first two equations for u and v. We now obtain the known formulas (1.18). Substituting these expressions into the formula for divergence X, we get

$$X = \frac{\partial w}{\partial z} + \frac{i}{a^2\sigma} F\left(\frac{p}{\bar{p}}\right), \tag{1.26}$$

where F - the tidal Laplacian. From (1.23) and (1.25) we exclude pressure

$$i\sigma w = -\frac{1}{i\sigma}\left(\frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dz} g w + g w_z + xgX - c^2 X_z\right) - g\frac{\rho}{\bar{\rho}}.$$

From here by means of (1.24) we exclude density

$$\sigma^2 w - g w_z = (x-1)\, gX - c^2 X_z. \tag{1.27}$$

Substituting into formula for divergence (1.26) pressure p, expressed through X and W formula (1.25) .

$$X = \frac{\partial w}{\partial z} + \frac{1}{\sigma^2 a^2} F\left(-c^2X + gW\right). \tag{1.28}$$

Now we get two equations (1.27) and (1.28) for X, w. Hence we exclude w, for the object of which we apply to both portions (1.28) operator $\sigma^2 - g\,\frac{\partial}{\partial z}$. We shall have

$$c^2 X_{zz} + \left[\frac{dc^2}{dz} - \varkappa g\right] X_z + \sigma^2 X + \frac{g}{\sigma^2 a^2} F\left(\frac{\sigma^2 c^2}{g} X - \beta X\right) = 0. \quad (1.29)$$

Thus, we have one equation for one variable quantity X. The given method for exclusion of variables is used in the theory of tides (see, for instance, Wilkes, 1949). But there the analysis is of somewhat simplified equations (approximation of quasistatistics). The simplification is that in the initial system of equations vertical accelerations are neglected, i.e. the left - hand portion of equation (1.3) is assumed to be zero. Correspondingly equation (1.29) is simpler with the use of quasistatistical approximation : it is short of two terms $\sigma^2 X$ and $\frac{\sigma^2 c^2}{g} X$.

It should be mentioned, that the only characteristic of the atmosphere's stratification, included in the obtained equation, is $c^2 = \varkappa R\bar{T}$. Instead of c^2 we shall bring in sometimes equivalent altitude of uniform atmosphere $H(z) = c^2/\varkappa g$.

For the sake of convenience we also carry out replacement of variables removing the term with first derivative in altitude, i.e. which brings the equation to a self-adjoint type

$$x = \int_0^z \frac{dz}{H(z)}, \quad x = e^{\frac{1}{2}x}\, y\,(\varphi,\ \theta,\ \chi).$$

These formulas could also be written in this way :

$$x = - \text{In} \frac{\bar{p}}{\bar{po}} , \ X = \sqrt{\frac{\bar{po}}{p}} \ y,$$

where po - is the pressure near the ground.

In the new variables we shall have

$$y_{xx} - \left(\frac{1}{4} - \frac{\sigma^2 H}{xg}\right) y + \frac{g}{\sigma^2 a^2} F\left(\frac{\sigma^2 H^2}{g} y - \frac{\beta H}{xg} y\right) = 0. \quad (1.30)$$

5. Division of variables.

In equation (1.30) we divide variables. For this we assume

$$y (\varphi, \theta, \chi) = \Psi (\varphi, \theta) y (x). \qquad (1.31)$$

For each of the multipliers we'll have a corresponding equation

$$F \Psi + \frac{a^2 \sigma^2}{gh} \Psi = 0, \qquad (1.32)$$

$$y + \left[- \frac{1}{4} + \frac{\sigma^2 H}{xg} (1 - \frac{xH}{h}) + \frac{\beta H}{xgh} \right] y = 0. \qquad (1.33)$$

Here a new constant h appeared as the invariable of variables division. This is not only a formal constant, it has a certain physical meaning, about which we shall speak further. As it will be shown, the solutions exist only with material values of h. But this constant could be both positive and negative. In equation (1.32) only the horizontal coordinates are included. Parameters included in this equation (earth radius and angular velocity), characterize horizontal structure of the atmosphere. In equation (1.33) only the vertical coordinate and equivalent altitude of the homogeneous atmosphere H, characterizing vertical stratification of atmosphere are included.

Thus, equation (1.32) is obtainable with all possible models of the atmosphere's vertical structure, whether it is isothermic atmosphere, or the standard atmosphere taken in this work, or a homogeneous ocean. Equation (1.32) is also obtainable even in the study of such a simple model as two-dimensional compressible film in the field of Coriolis forces. In the last two cases equation (1.33) is not present ; h does not appear anymore in division of variables. In the case of a homogeneous ocean h means simply the depth of this ocean, and in the case of two-dimensional film $h = c^2/g$, where c - adiabatic velocity of sound , $c^2 = \chi\ \bar{p}/\bar{\rho}$.

Thus, while we are dealing with horizontal structure of oscillations, the atmosphere could be replaced by a homogeneous ocean of depth, h natural oscillation of which has the same frequency and horizontal structure $\Psi(\varphi,\ \theta)$, as the analyzed oscillation of atmosphere. This is why h bears the name of dynamically equivalent depth for the given natural oscillation in distinction from statically equivalent depth H(z).

If we are concerned only with the vertical structure of oscillations, we apply equation (1.33), and the adopted horizontal model, i.e. parameters ω and a will be found to make no difference. In particular, it could be assumed, that the Earth is flat ($a \to \infty$) and does not rotate ($\omega \to \infty$). It has already been said, that at $\omega \to 0$ operator F converts into Laplacian on a single sphere, and at $a \to \infty$ the sphere itself converts into plane; $\frac{1}{a^2}$ F in this case tends to Laplacian on a flat plane. Equation (1.32) changes into equation

$$\Delta\Psi + \frac{2}{gh}\ \Psi = 0, \tag{1.34}$$

which is an ordinary two-dimensional wave equation on a flat plane. Phase velocity of these waves is $\sqrt{gh}$, since is frequency. Thus, the meaning of parameter h is clarified from one more side, from the view-point of its role in equation (1.33). If the vertical structure of atmosphere and the given oscillation y(x) is preserved, the atmosphere assumed to be horizontally flat and non-rotatory, and the solution to be sinusoidal, then h is equal to a depth, at which phase velocity.

$$C\phi = \sqrt{gh}. \tag{1.35}$$

It should also be additionally mentioned, that the model of flat non-rotating Earth represents very well the horizontal structure of oscillations of not very large horizontal scale, when the curvature of the Earth and its rotation have no appreciable effect.

Equations (1.32) and (1.33) should be resolved under certain limits. As regards the equation (1.32), the question here is quite clear. This equation is on a closed sphere, and the requirement is only the regularity of the solution, since in this case there are no boundaries. Equation (1.33) requires two boundary conditions, as this equation is of second order in x. One of these conditions is set on the Earth's surface. Actually, on the surface of the Earth it is necessary to have w equal to zero, and hence to obtain the required boundary condition for y (it will be obtained further). As regards the condition on infinity, it should be mentioned, that since $H \longrightarrow \infty$ at $x \longrightarrow \infty$, the solutions of equation (1.33), as can be easily proved, approach very quickly (faster than the exponent) towards infinity, except one, approaching just as fast as the exponent towards zero. If limits for solutions are set at infinity, then out of all the solutions only one remains-transient. This is the condition that will be adopted.

Now let us return to boundary condition on the surface of the Earth. Let us also find in the form, where the variables will be divided.

$$w(\varphi, \theta, x) = e^{\frac{1}{2}x}\,\Psi(\varphi, \theta)\,\mathcal{W}(x). \qquad (1.36)$$

Then equations (1.27) and (1.28) will be :

$$(\sigma^2 - \frac{g}{2H})\mathcal{W} - \frac{g}{H}\mathcal{W}' = (\frac{x}{2} - 1)\,gy - xgy',$$

$$(-\frac{1}{h} + \frac{1}{2H})\mathcal{W} + \frac{1}{H}\mathcal{W}' = (1 - \frac{xH}{h})\,y. \qquad (1.37)$$

Resolving this system as algebraical in relation to w and w', we find w

$$\left(\sigma^2 - \frac{g}{h}\right) w = -xg \left[y' + \left(\frac{H}{h} - \frac{1}{2}\right) y \right]. \qquad (1.38)$$

Out of this relation it is possible to find w, after y has been found. As mentioned previously, it is required, that on the surface of the Earth $w = 0$. For those solutions, in which $\sigma^2 - \frac{g}{h} \neq 0$, this is equivalent to boundary condition

$$y' + \left(\frac{H}{h} - \frac{1}{2}\right) y = 0 \quad \textit{and} \quad x = 0. \qquad (1.39)$$

As shown by (1.38), for solutions, in which $^2 - \frac{g}{h} \neq 0$, the limit of y is equivalent to the limit of .

Now our problem is reduced to the following. There are two equations : (1.32) and (1.33). These contain two parameters σ and h, which have to be selected in such a way that our equations to have solutions, meeting boundary the conditions. Each of the equations could be studied separately. If some value is fixed for one of the parameters, the result will be a problem on eigen values of the second. Varying the values of the first parameter, we find new eigen values of the second. Thus, on the plane of these parameters we get a set of curves, which we shall name the self-curves of the equation. In the same way it is possible to plot self-curves of the second equation. At the intersection points of self-curves of the first and second equations we will get the required eigen values σ and h, at which both the equations will have the solutions.

6. Peckeris solution.

A unique case occurs at $\sigma^2 = g/h$. Uptil now the solution was of equation for divergence X, or for y. After the y has been determined, the values should be obtained of the remaining quantities, including w. As the equation (1.38) shows, that at $\sigma^2 = g/h$ there would be solution w, it is necessary, that

$$y' + \left(\frac{H}{h} - \frac{1}{2}\right) y \quad 0. \tag{1.40}$$

Thus, besides the fact, that y should satisfy differential equation of second order (1.33), it should also be the solution for the first order equation (1.40). If these equations were independent, then, as a rule, there would not be these solutions. But here, it seems, the following interesting state occurs. For any σ and h, bound by relation $\sigma^2 = g/h$, there is indeed a soultion for (1.33), which satisfiês all the set boundary conditions and, moreover, the first order equation (1.40). To be more exact, it will be shown, that at $\sigma^2 = g/h$ any solution of the first order equation satisfies also the second order equation.

We differentiate (1.40) and instead of y we substitute its expression from (1.40). The result is

$$y'' + \frac{H'}{h} y - \left(\frac{H}{h} - \frac{1}{2}\right)^2 y = 0,$$

which coincides with equation (1.33) with acounting for $\sigma^2 = g/h$. Since it is assumed, that H increases at infinity, the soultion of (1.40) at infinity quickly vanishes. Therefore, the boundary condition at infinity is found to be fulfilled. Boundary condition (1.39) on the surface of the Earth is automatically satisfied as a result of equation (1.40).

It would seem, that from the evidence presented, it is possible to deduce, that any pair of σ and h values, bound by relation σ^2=g/h, is a pair of eigen values, i.e. that determination has been made of one of the equation (1.33) self-curves - curve σ^2= g/h. In this case it is found, this curve is absolutely independent of H, i.e. on stratification, if only H would increase on infinity (sufficient, if it does not decrease). However, it may be assumed, that the obtained solution is extraneous, not corresponding to the physical set up of the problem. In fact, after the y value has been determined, it is necessary to determine the values of the remaining unknown quantities, primarily w. Equation (1.38) is more unsuitable for this object, as both the portions get converted into zero. Application should be made of the system of equations (1.37), both equations of which happen to be in our case identical. For determination of w solution has to be made of any of these differential equations with respect to w has to be solved. In this case it is necessary to satisfy boundary condition w = 0 at x = 0. If solution of homogeneous equation

$$w' - \left(\frac{H}{h} - \frac{1}{2} \right) w = 0$$

is denoted by w_0 (apparently, $w_0 = \exp \int_0^x (H/h - 1/2)dx$). then

$$w = w_0 \int_0^x H \left(1 - \frac{xH}{h} \right) \frac{y}{w_0} \, dx,$$

or

$$w = e^{\int_0^x \left(\frac{H}{h} - \frac{1}{2}\right) dx} \int_0^x H \left(1 - \frac{xH}{h}\right) e^{2 \int_0^x \left(\frac{1}{2} - \frac{H}{h} \right) dx} dx.$$

It is easy to see, that with the increment of altitude x at all values of h, except individual, for which

$$\int_0^{\infty} H \left(1 - \frac{xH}{h}\right) e^{2\int_0^x \left(\frac{1}{2} - \frac{H}{h}\right) dx} dx = 0, \qquad (1.41)$$

this solution quickly tends to infinity.

Thus, for every h solution for y is plotted, which satisfies the condition of limit at infinity, but in this case w happens to be limitless and so quickly increasing, that the energy of oscillations is found to be infinite. This means, that these oscillations cannot be excited. As will be shown in chapter 6, the completeness of the system of fundamental functions takes place, if solution $\sigma^2 = g/h$ is not taken into account, i.e. arbitrary solution could be expanded into linear combination of the remaining fundamental functions. In view of this we shall assume, that solution $\sigma^2 = g/h$ has no physical meaning.

We remind, that at $\sigma^2 \neq g/h$ the damping of y at infinity was equivalent to damping of w. For solutions under analysis at damping y the vertical velocity quickly increases. If from the initial condition we set the limit of w, these solutions become superfluous.

We discussed in such detail one particular solution due to the following circumstance. As we will see further, this solution results in a whole class of waves, the existence of which was first discovered by Peckeris (1948) with the use of isothermic model of atmosphere. Here it also shows, firstly, that the curve $\sigma^2 = g/h$ is a self-curve of equation (1.33) for any stratification, and secondly, that not the less corresponding solutions should not be taken into account, as they are, apparently, devoid of physical meaning.

It should be mentioned, that $\sigma^2 = g/h$ happened to be self-curve only due to assumption, that H at infinity does not decrease. Otherwise it may not be so.

In the next two chapters a theoretical solution will be proposed for the equation (1.32) for horizontal component, in chapter 4 - theoretical solution of equation (1.33) for vertical component.

Chapter. 2.

LAPLACIAN EQUATION OF THE THEORY OF TIDES.

1. Laplace's equation. Integral relations.

In the present chapter a step-by-step study of equation (1.19) will be made. In spite of the fact, that, beginning from Laplace, this equation was analyzed by many investigators, its theory cannot be taken as completed.

First of all without difficulty it is possible to separate longitude φ, assuming, that $\Psi(\varphi, \theta) = e^{is\varphi}\Psi(\cos\theta)$, $s = 0, \pm 1, \pm 2 \ldots\ldots$ Then $\Psi(\cos\theta)$ satisfies the equation

$$F_s\,\psi + \frac{a^2\sigma^2}{gh}\,\psi = 0,$$

where operator F_s - operator F, in which $\partial/\partial\varphi$ are everywhere replaced by is. In more detail

$$\left\{ \frac{f^2}{\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{\sin\theta}{f^2-\cos^2\theta}\left(\frac{\partial}{\partial\theta} + \operatorname{ctg}\theta\,\frac{s}{f}\right)\right] + \right.$$

$$\left. + \frac{f^2}{f^2-\cos^2\theta}\left[-\operatorname{ctg}\theta\,\frac{s}{f}\frac{\partial}{\partial\theta} - \frac{s^2}{\sin^2\theta}\right]\right\}\psi + \frac{a^2\sigma^2}{gh}\,\psi = 0. \quad (2.1)$$

Same as σ or $f = \sigma/2$, s could be positive or negative. Equation (2.1) includes only s/f, i.e. if there is a solution of Ψ for some s and f, then exactly the same solution is available for $-s$ and $-f$. In other words, the solutions are always available in pairs

$$e^{\pm i(\sigma t + s\varphi)} \psi(\cos\theta).$$

Assuming for discretness, that s is positive, and σ could have sign. The actual solutions are obtained as real and imaginary parts of a complex solution. They will have the form $\cos(\sigma t + s\varphi)\psi(\cos\theta)$ and $\sin(\sigma t + s\varphi)\psi(\cos\theta)$. If $s/f > 0$, the waves are directed east to west, if $s/f < 0$, west to east.

Bringing in a new variable $\mu = \cos\theta$. We get

$$\left[(1-\mu^2)\frac{d}{d\mu} + \frac{s\mu}{f}\right] \frac{1}{f^2-\mu^2} \left[(1-\mu^2)\frac{d}{d\mu} - \frac{s\mu}{f}\right] \psi(\mu) =$$

$$= \left[\frac{s^2}{f^2} - (1-\mu^2)\gamma\right] \psi(\mu), \quad \gamma = \frac{4a^2\omega^2}{gh}. \qquad (2.2)$$

Equation (2.1) or (2.2) is also called Laplace's equation of the theory of tides, and its solutions - Hough's functions.

Equation (2.2) has special points : firstly, those are ends of segment $\mu = \pm 1$ (or $\theta = 0, \pi$), secondly, its value is $\mu = \pm f$ (of latitude θ for which $\cos\theta = \pm f$ bear the name of critical latitudes). If the solutions are expanded into series within the range of special points $\mu = \pm 1$, it can be discovered, that at each of these points there is one solution, which is the product $(1-\mu^2)^{\frac{s}{2}}$ by analytical function, and one solution with logarithmic branching. As regards the critical latitude, there could have been danger here, if the solutions were discontinuous at this point (or had discontinuous derivative). It would have been necessary to seek conditions at discontinuity see Brilloin work, (1932), devoted to overcoming the pointed out difficulty . Actually t is difficulty is fictitious, because in spite of the discontinuity factor of the equation, it is easy to show by expanding solutions at these points into series, that they are continuous and even analytical within the region of these special points. There is one non-trivial solution, which converts at a special point jointly with its derivative into zero, and for other solutions the relation is fulfilled.

$$(1 - f^2)\psi' (\pm f) - s\psi(\pm f) = 0.$$

Anyhow the existence of critial latitudes have still one complication, which we shall discuss later.

Besides, the azimuthal wave number s, equation (2.2) includes two parameters : oscillation frequency f and the parameter y, which will be used in this chapter instead of dynamically equivalent depth h.

Complications, connected with formal presence of special points, could be avoided, if following Eckart (1960) equation of the second order is replaced by a system of two first order equations, introducing new function. ζ

$$\left[(1- \mu^2) \frac{d}{d\mu} - \frac{s\mu}{f}\right]\psi (\mu)= (f^2 - \mu^2) \zeta(\mu),$$

$$\left[(1-\mu^2)\frac{d}{d\mu}+\frac{s\mu}{f}\right]\zeta(\mu)=\left[\frac{s^2}{f^2}-(1-\mu^2)\gamma\right]\psi\quad(\mu). \quad (2.3)$$

The first one is used for determining the functions ζ, then the equation (2.2) gets reduced to the second equation only.

Now we prove, that for any real value of f the eigen values of y are always real. With this object we multiply the first equation by $\zeta^*(\mu)$ (the asterisk means complex conjugate), the second equation is replaced by the complex conjugate and multiply by ψ (μ), we add these equations, divide by $1-\mu^2$ and integrate by μ from -1 to 1. Herewith it should be mentioned, that although the points at the ends are special, the integration is possible, since the solutions convert into zero because of $(1-\mu^2)^{\frac{s}{2}}$. We shall get

$$0=\int_{-1}^{1}\frac{f^2-\mu^2}{1-\mu^2}|\zeta|^2\,d\mu+\int_{-1}^{1}\left[\frac{s^2}{f^2(1-\ ^2)}-\gamma^*\right]|\psi|^2\,d\mu,$$

or

$$\gamma^*=\frac{\int_{-1}^{1}\frac{f^2-\mu^2}{1-\mu^2}|\zeta|^2\,d\mu+\int_{-1}^{1}\frac{s^2}{f^2(1-\mu^2)}|\psi|^2\,du}{\int_{-1}^{1}|\psi|^2\,d\mu}. \quad (2.4)$$

Hence it follows, that y is real. From this formula it is possible to make one more important deduction. Eigen value y, generally speaking, could be both positive and negative. But for $f^2\geqslant 1$, i.e. for periods less than semidiurnal y is always positive.

In exactly the same way it is possible to prove one more important relation-orthogonality of fundamental functions ψ_1 and ψ_2 for various y (at one and the same f). With this object we write down all the equations which satisfy $\psi_1, \zeta_1, \psi_2, \zeta_2$:

$$\left[(1- \mu^2) \frac{d}{d\mu} - \frac{s\mu}{f} \right] \psi_1 = (f^2 - \mu^2) \zeta_1 ,$$

$$\left[(1- \mu^2) \frac{d}{d\mu} + \frac{s\mu}{f} \right] \zeta_1 = \left[\frac{s^2}{f^2} - (1- \mu^2) y_1 \right] \psi_1 ,$$

$$\left[(1- \mu^2) \frac{d}{d\mu} - \frac{s\mu}{f} \right] \psi^*_2 = (f^2 - \mu^2) \zeta^*_2,$$

$$\left[(1- \mu^2) \frac{d}{d\mu} + \frac{s\mu}{f} \right] \zeta^*_2 = \left[\frac{s^2}{f^2} - (1- \mu^2) y_2 \right] \psi^*_2.$$

The last two equations are replaced here by their complex conjugate. We multiply them respectively by ζ_2^*, ψ_2^*, ζ_1, ψ_1, from the sum of the first and the fourth equation we substract the sum of the second and third, divide the result by $1 - \mu^2$ and integrate from $- 1$ to 1. At $y_1 \neq y_2$ we shall have

$$\int_{-1}^{1} \psi_1 \psi_2^* \, d\mu = 0. \tag{2.5}$$

2. Fundamental curves. Asymptotics at low y.

Thus, setting arbitrarily the values of parameter f and determining eigen values of parameter y, we get in plane (f, y) a set of fundamental curves. Simple reasoning permits to fix, that these curves cannot intersect. Indeed, let us assume a different case. Let it be, that at f, approximating certain value f_o, two eigen values γ_1 and γ_2 merge into one, γ_o. In this case the fundamental functions either strive towards two different, linearly independent functions, or to one and the same. The first is eliminated by the condition, that there is only one solution, which at the end of the segment converts into zero, whereas all the others convert into infinity. Thus, there cannot be two linearly independent solutions, regular at the end of segment. The spectrum of eigen values is simple.

Now let both the fundamental functions strive towards one and the same function, i.e. $\psi_1 \psi_o$, $\psi_2 \rightarrow \psi_o$. On one hand it could be assumed that ψ_1, ψ_2 and hence their limits as standard, $\int_{-1}^{1} \psi_o \psi_o^* \, d\mu = 1$. On the other hand, ψ_1 and ψ_2 are orthogonal toward each other [see formula (2.5)]. Then within the limit $\int_{-1}^{1} \psi_o \psi_o^* \, d\mu = 0$. The obtained contradiction proves, that with continuous variation of parameter f there cannot be the merging of eigen values and fundamental functions, i.e. in the language of algebra, there cannot be formation of Jordan cage. This is a simple result of self-adjoint or the orthogonality of fundamental functions.

Let us carry out now a more detailed investigation of the shape of natural curves and their disposition in plane. Let us analyse first their ultimate behavior at some extreme values of parameters. We start from two simple cases, which were well known at least to Margules (1893) and Hough (1898). The first of these cases pertains to asymptotic behavior of self-curves for high frequencies, i.e. at high f values. In this case the system (2.3) could be approximately written as :

$$(1 - \mu^2) \frac{d \psi}{d \mu} = f^2 \xi ,$$

$$(1- \mu^2) \frac{d \xi}{d \mu} = \left[\frac{s^2}{f^2} - (1 - \mu^2) \gamma \right] \psi .$$

From here we may exclude the auxiliary variable ψ , thereafter obtaining for

$$(1 - \mu^2) \psi'' - 2 \mu \psi' - \frac{s^2}{1- \mu^2} \psi + f^2 \gamma \psi = 0.$$

This is nothing else, but Legendre equation. It has uniform solutions at $f^2 y = n (n + 1)$. In this case $\psi = P^s_n (\mu)$. Thus, we get the following asymptotic formulas :

$$|f| \sim \sqrt{\frac{n (n + 1)}{\gamma}} ,$$

$$\psi \sim P^s_n (\mu),$$

$$n = S , S + 1, \ldots \ldots \qquad (2.6)$$

Actually these asymptotics were obtained in chapter 1, when as a result of ultimate transition $\omega \longrightarrow 0$ equation (1.34) was deduced. True, there was one more ultimate transition, $a \longrightarrow \infty$. But if the last transition i is not implemented, the Δ operator will become Laplacian on a sphere of a radius, eigen values of which are $- n (n + 1) / a^2$, and the result is asymptotic (2.6).

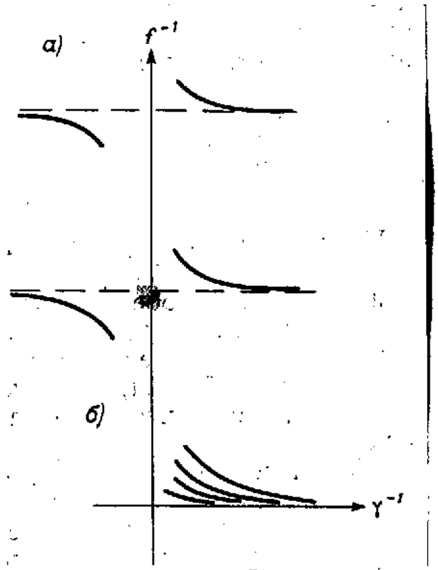

Fig. 2.1. Asymptotic behavior of self-curves of Laplace tidal equation at high y^{-1}.

a) $f^{-1} = \dfrac{n(n+1)}{s}$,

b) $f^{-1} = \dfrac{1}{\sqrt{n(n+1)}} \cdot \dfrac{1}{\sqrt{\gamma - 1}}$.

To illustrate the obtained results we shall use the plane (f^{-1} , y^{-1}). This is convenient, because f^{-1} is proportional to period, and γ^{-1} - to dynamically equivalent depth. Self-curves of equation (2.6) are shown in Fig. 2.1. by lines, asymptotically approaching the abscissae at $\gamma^{-1} \to \infty$. Such is the first of the known asymptotic formulas. But, it turns out, that at $\gamma^{-1} \to \infty$ all curves are not approaching the abscissae. The second known asymptotic curve pertains to the case of high $^{-1}$ and finite f. In this case the system could approximately be written as :

$$\left[(1-\mu^2)\frac{d}{d\mu} - \frac{s\mu}{f}\right]\psi = (f^2 - \mu^2)\zeta ;$$

$$\left[(1-\mu^2)\frac{d}{d\mu} + \frac{s\mu}{f}\right]\zeta = \frac{s^2}{f^2}\psi .$$

Here it is easier to eliminate ζ instead of ψ ,. For ζ we get equation

$$(1-\mu^2)\zeta - 2\mu\zeta' - \frac{s^2}{1-\mu^2}\zeta + \frac{s}{f}\zeta = 0.$$

The result is again the Legendre polynomial, regular solutions of which are available, if $s/f = n(n+1)$. In this case $\zeta = P_n^s(\mu)$. As regards the ψ, it could be shown as a linear combination of two adjoined Legendre polynomials. We have the following asymptotic concept :

$$f \sim \frac{s}{n(n+1)} ,\ \psi \sim \frac{n^2(n-s+1)P_{n+1}^s + (n+1)^2(n+s)P_{n-1}^s}{2n+1}, \qquad (2.7)$$

$$n = s,\ s+1,\ \ldots..$$

Thus, as the first asymptotic term we get the equations of horizontal asymptotes. Fig.2.1 also shows these asymptots. In para 4 both the given asymptotes will be made more exact.

The following condition should be mentioned. While the asymptote (2.6) does not depend on the sign of F(at least the given chief term), formula (2.7) gives positive values of f, i.e. pertains to waves, propagating east to west.

Thus, according to behavior at $y^{-1} \to \infty$ all the self-curves of Laplacian tidal equation are divided into two groups-some approach to abscissae while, others step out on horizontal asymptotes,distinct from the abscissae axis. Following Hough we shall call these curves of the first and second nature in relation to high γ^{-1}. With one and the same fixed s there is a topmost curve among those of the first type, which corresponds to n=s, and they accumulate toward the abscissae. Among the curves of second type the lowermost also corresponds to n = s.

We are speaking here only of positive values of parameter γ^{-1}. If we apply negative values, then first of all it will be noted, that there cannot be any asymptote at $/f/ \to \infty$, since there is no existence of γ^{-1} negative values at $/f/ \geqslant 1$, as pointed out in the preceding para. On the contrary, asymptotics curves (2.7) are totally retained even at negative γ^{-1}, i.e. at $\gamma^{-1} \to -\infty$. The curves approach each of these horizontal asymptote from both sides at $\gamma^{-1} \to \infty$ and at $\gamma^{-1} \to -\infty$.

In chapter 3 we shall obtain new asymptotes, at low γ^{-1}. It will be found, that the activity region of both the asymptotes jointly cover practically the whole range of values and depict wholly all the self-curves

3. <u>Conversion to type convenient for application of Galerkin's method.</u>

We pass on to the plotting of algorithm for calculating self-curves of the Laplacian tidal equation. First of all we note, that the fundamental function asymptotically is either Legendre function $P_n^s(\mu)$, or a linear combination of two such functions. It is also natural for the remaining parameter values, when asymptotes are inapplicable, to seek solutions in the form of linear combination of some number of these functions P^s_{n1}, P^s_{n1+1}, ..., P^s_{n1+N}, i.e. to approximate differential equation by a finite algebraical system, or in other words, to apply Galerkin method. As a preliminary the system of equations should be converted to a more convenient form. Why the system (2.3) is inconvenient, could be explained in the following way. Differential operator, included in our equations, $(1 - \mu^2) d/d\mu$, applied to Legendre polynomial P_n^s, same as the multiplication of this function by μ, is replaced by its linear combination P^s_{n-1} and P^s_{n+1}. But, unfortunately, if the equations are used in the form, in which they are written, the Legendre functions would have to be multiplied also by μ^2, and this will result in the appearance of functions P^s_{n-2} and P^s_{n+2}.

This will make the system of equations considerably more complex and bulky. It has to be converted so, that there is no multiplication by μ^2. We denote

$$L = (1- \mu^2) \frac{d^2}{d\mu^2} - 2\mu \frac{d}{d\mu} - \frac{s^2}{1 - \mu^2}$$

(this is nothing, but the operator in the left portion of Legendre equation). The result of this second order operator's effect on P^s_n is very simple

$$LP_n^s = -n(n+1)P_n^s .$$

Let us apply to the second of (2.3) equations differential operator $(1-\mu^2)\,d/d\mu - s\mu/f$, and then substitute $\left[(1-\mu^2)\,d/d\mu - s\mu/f\right]\psi$ from the first equation. We get

$$\left[(1-\mu^2)\frac{d}{d\mu}-\frac{s\mu}{f}\right]\left[(1-\mu^2)\frac{d}{d\mu}+\frac{s\mu}{f}\right]\zeta =$$

$$=\frac{s^2(f^2-\mu^2)}{f^2}\zeta-\left[(1-\mu^2)\frac{d}{d\mu}-\frac{s\mu}{f}\right](1-\mu^2)\gamma\psi .$$

By expanding in the left portion the product of operators and dividing equation by $1-\mu^2$, we shall have

$$\left[\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}+\frac{s}{f}-\frac{s^2\mu^2}{f^2(1-\mu^2)}\right]\zeta =$$

$$=\frac{s^2(f^2-\mu^2)}{f^2(1-\mu^2)}\zeta-\frac{d}{d\mu}(1-\mu^2)\gamma\psi+\frac{s\mu}{f}\gamma\mu ,$$

or

$$\left(L+\frac{s}{f}\right)\zeta = \gamma\left[\left(\frac{s}{f}+2\right)\mu-(1-\mu^2)\frac{d}{d\mu}\right]\psi .$$

This equation will be the first of the two, which compose our working system.

The second equation is obtained in the same way, but now to the first equation of (2.3) we apply operator $(1-\mu^2)\,d/d\mu + s\mu/f$, and the expression $\left[(1-\mu^2)\,d/d\mu + s\mu/f\right]\zeta$ we substitute from the second equation. Then we get

$$\left[\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu} - \frac{s}{f} - \frac{s^2\mu^2}{f^2(1-\mu^2)} \right]\psi =$$

$$= \frac{f^2-\mu^2}{1-\mu^2}\,\frac{s^2}{f^2}\psi - (f^2-\mu^2)\gamma\psi - 2\mu\zeta \quad .$$

The term $(f^2 - \mu^2)\gamma\psi$ we convert in the following way.

We replace it by $(1 - \mu^2)\gamma\psi + (f^2 - 1)\gamma\psi$ and the first of the two terms we again express from the second equation. We shall have

$$\left[\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu} - \frac{s}{f} - \frac{s^2}{1-\mu^2} \right]\psi =$$

$$= \left[(1-\mu^2)\frac{d}{d\mu} + \frac{s\mu}{f} \right]\zeta - \frac{s^2}{f^2}\psi - 2\mu\zeta - (f^2-1)\gamma\psi \quad .$$

By conversion, we finally obtain

$$\left[L - \frac{s}{f} + \frac{s^2}{f^2} + (f^2-1)\gamma \right]\psi = \left[\left(\frac{s}{f}-2\right)\mu + (1-\mu^2)\frac{d}{d\mu} \right]\zeta \quad .$$

Now we write jointly both the obtained equations :

$$\left(L + \frac{s}{f} \right) = \gamma\left[\left(\frac{s}{f}+2\right)\mu - (1-\mu^2)\frac{d}{d\mu} \right]\psi ,$$

$$\left[L - \frac{s}{f} + \frac{s^2}{f^2} + (f^2-1)\gamma \right]\psi = \left[\left(\frac{s}{f}-2\right)\mu + (1-\mu^2)\frac{d}{d\mu} \right]\zeta \quad . \tag{2.8}$$

This system has a higher order (fourth), but due to above indicated reason it is found to be more convenient for the application of Galerkin's method. Moreover, as will be seen from the next chapter, this system will help to find the asymptotics of characteristic curves at low γ^{-1}. However, there are some questions, as to whether the system (2.8), being the result of system (2.3) contains extraneous resolutions ? After all it was obtained by rising the order of system. Having found eigen values of this system's parameters, do we have the assurance, that they pertain to the initial (2.3) system ? In the " Supplement " to the present chapter this question will be investigated. Here we only formulate the result. It is found, that extraneous solutions are indeed present. The system (2.8) has regular solutions at all whole values s/f and at any γ , i.e. there is the appearance of characteristic curves-horizontal straight lines, passing through all the points of axis s/f. At the same time they are not the characteristic curves of system (2.3). Other system does not have extraneous solutions system (2.8). Now we can safely utilize this system.

Here is another form of the Laplacian tidal equation. This form is investigated in the Yaglom's article (1953). This work investigates the example, mentioned by us of the two-dimensional flow on the surface of a sphere. The field velocity is shown through current function and potential. The system of equations in Yaglom's article with slightly changed denotations is as follows

$$if L\Psi + is\Psi + L\Phi + (1-\mu^2)\frac{d\Phi}{d\mu} = 0, \qquad (2.9a)$$

$$\mathrm{if}L\bar{\Phi} + \mathrm{is}\Phi - \mu L\Psi - (1-\mu^2)\frac{d\Psi}{d\mu} + L\pi = 0, \tag{2.9b}$$

$$\mathrm{if}\,\gamma\pi + L\Phi = 0, \tag{2.9c}$$

where $\tilde{\Psi}$, Φ and π - respectively current function, potential and pressure, or more exactly their portions, depending on polar angle. Other notations are the same as above. Let us fix correspondence between the tidal Laplacian equation and the system (2.9). Let ψ , ξ be solutions of Laplace's equation (2.3) and hence also of (2.8). We determine functions Ψ , Φ , and in the following way. Assuming

$$\pi = \psi . \tag{2.10}$$

Assuming further, that Φ is determinable from equation (2.9c), and Ψ from equation

$$\xi = \frac{s}{f}\Psi + \frac{1-\mu^2}{\mathrm{if}}\frac{d\Phi}{d\mu} . \tag{2.11}$$

Then these functions satisfy the whole system (2.9). This fact is proved in the " Supplement " to this chapter. The meaning of the system (2.9) is the same as of system (2.8) : it is redifferentiated system of Laplace's equations. When the system (2.9) was being obtained in Yaglom's article an excessive differentiation was carried out in transition from velocities to equation for vortex and divergence. It will be shown, that sometimes it is convenient to use the system (2.8), sometimes system (2.9).

4. Calculation of characteristic curves of Laplace's equation of the theory of tides.

We shall seek solutions in the form of series of fixed and adjoined Legendre polynomials

$$\psi = \sum_{n=s}^{\infty} a_n \sqrt{\frac{2n+1}{2}\frac{(n-s)!}{(n+s)!}}\, P_n^s ,$$

$$\zeta = \gamma \sum_{n=s}^{\infty} b_n \sqrt{\frac{2n+1}{2}\frac{(n-s)!}{(n+s)!}}\, P_n^s . \qquad (2.12)$$

We substitute these series into system (2.8), using in this case the known recurrent relations :

$$\mu P_n^s = \frac{n-s+1}{2n+1} P_{n+1}^s + \frac{n+s}{2n+1} P_{n-1}^s ,$$

$$(1-\mu^2)\frac{dP_n^s}{d\mu} = -\frac{n(n-s+1)}{2n+1} P_{n+1}^s + \frac{(n+1)(n+s)}{2n+1} P_{n-1}^s ,$$

$$LP_n^s = -n(n+1)\, P_n^s .$$

Equating factors at P_n^s with similar indices, we get :

$$\left[-n(n+1)+\frac{s}{f}\right] b_n = \sqrt{\frac{(n-s)(n+s)}{(2n-1)(2n+1)}}\left(\frac{s}{f}+n+1\right)a_{n-1} +$$

$$+\sqrt{\frac{(n-s+1)(n+s+1)}{(2n+1)(2n+3)}}\left(\frac{s}{f}-n\right)a_{n+1} ,$$

$$\left\{ \left[- n(n+1) - \frac{s}{f} + \frac{s^2}{f^2} \right] \gamma^{-1} + f^2 - 1 \right\} \quad a_n =$$

$$= \sqrt{\frac{(n-s)(n+s)}{(2n-1)(2n+1)}} \left(\frac{s}{f} - n - 1 \right) b_{n-1} +$$

$$+ \sqrt{\frac{(n-s+1)(n+s+1)}{(2n+1)(2n+3)}} \left(\frac{s}{f} + n \right) b_{n+1}. \tag{2.13}$$

From this system it is possible to eliminate either a_n or b_n. Eliminating b_n we get for a_n a differential equation

$$\gamma^{-1} a_n = L_{n-2} a_{n-2} + M_n a_n + L_n a_{n+2}, \quad n \geqslant s, \tag{2.14}$$

where

$$M_n = \frac{f^2 - 1}{\left(-\frac{s}{f} + n \right)\left(\frac{s}{f} - n - 1 \right)} +$$

$$+ \frac{(n-s)(n+s)\left(\frac{s}{f} - n + 1 \right)}{(2n-1)(2n+1)\left(\frac{s}{f} + n \right) \left[\frac{s}{f} - n(n-1) \right]} +$$

$$+ \frac{(n-s+1)(n+s+1)\left(\frac{s}{f} + n + 2 \right)}{(2n+1)(2n+3)\left(\frac{s}{f} - n - 1 \right) \left[\frac{s}{f} - (n+1)(n+2) \right]},$$

$$L_n = \frac{\sqrt{(n+s+1)(n+s+2)(n-s+1)(n-s+2)}}{(2n+3)\sqrt{(2n+1)(2n+5)} \left[\frac{s}{f} - (n+1)(n+2) \right]}.$$

we wont write equation for b_n , as we wont have to use it further. Instead we shall write differential equations, which could be obtained from the system of equations (2.9). If it is assumed, that

$$\Psi = \sum_{n=s}^{\infty} \tilde{b}_n \sqrt{\frac{2n+1}{2} \frac{(n-s)!}{(n+s)!}}\, P_n^s ,$$

$$\Pi = \sum_{n=s}^{\infty} a_n \sqrt{\frac{2n+1}{2} \frac{(n-s)!}{(n+s)!}}\, P_n^s , \tag{2.15}$$

then apparently,

$$\Phi = if \sum_{n=s}^{\infty} \frac{\tilde{an}}{n(n+1)} \sqrt{\frac{2n+1}{2} \frac{(n-s)!}{(n+s)!}\, P_n^s}$$

and

$$\left[\frac{s}{f} - n(n+1)\right] \tilde{b}_n = \gamma \left(\sqrt{\frac{(n-s)(n+s)}{(2n-1)(2n+1)}} \frac{n+1}{n} \tilde{a}_{n+1} + \right.$$

$$\left. + \sqrt{\frac{(n-s+1)(n+s+1)}{(2n+1)(2n+3)}} \frac{n}{n+1} \tilde{a}_{n+1} \right) ,$$

$$\left[N(n+1) + \frac{sf}{n(n+1)} - f^2\gamma\right] \tilde{a}_n =$$

$$= \sqrt{\frac{(n-s)(n+s)}{(2n-1)(2n+1)}} (n-1)(n+1)\tilde{b}_{n-1} +$$

$$+ \sqrt{\frac{(n-s+1)(n+s+1)}{(2n+1)(2n+3)}} n(n+2)\tilde{b}_{n+1}. \tag{2.16}$$

This system is reduced to one equation

$$\tilde{L}_{n-2}\,\tilde{b}_{n-2} + \tilde{M}_n\tilde{b}_n + \tilde{L}_n\tilde{b}_{n+2} = 0,\ n \geqslant s, \tag{2.17}$$

where

$$\tilde{M}_n = \left[n(n+1) - \frac{s}{f} \right] + \frac{(n-1)^2(n+1)^2(n-s)(n+s)}{(2n-1)(2n+1)\left[(n-1)^2 n^2 \gamma^{-1} + sf - (n-1)nf^2\right]} +$$

$$+ \frac{n^2(n+2)^2(n-s+1)(n+s+1)}{(2n+1)(2n+3)\quad (n+1)^2(n+2)^2 \quad {}^{-1} +},$$

$$L_n = \frac{\sqrt{(n-s+1)(n-s+2)(n+s+1)(n+s+2) \times n(n+1)(n+2)(n+3)}}{(2n+3)\quad (2n+1)(2n+5)\quad (n+1)^2(n+2)^2 \quad {}^{-1} + sf - (n+1)(n+2)f^2 \,]}.$$

System (2.16) is similar in form to (2.13). This system is investigated in the mentioned Yaglom's article. Generally, everything pertaining to system (2.9) and its consequences is borrowed from this article.

Before, explaining, in what way equation (2.14) could be used for calculating eigenvalues of γ , we return to previously obtained asyymptotic formulas (2.6) and (2.7) and find their exact definitions. One of the possible methods for finding asymptotes at $\gamma^{-1} \to \infty$ consists in expanding all values a_n , b_n and f into series according to the order of γ and equating of terms, containing in the same order, from both sides of equations (2.13) or (2.16). In this way it is possible to obtain, for instance, first exact definition of asymptotic formula (2.7)

$$\frac{s}{f} \sim n(n+1) + \gamma\left[\frac{(n-s)(n+s)(n+1)^2}{(2n-1)(2n+1)\,n^2} + \right.$$

$$\left. + \frac{(n+1-s)(n+1+s)\,n^2}{(2n+1)(2n+3)(n+1)^2}\right] . \qquad (2.18)$$

In the same way it is possible to obtain also the following terms of expansion, although there is no special need for that. This method is described in detail in the author's article (1961). We demonstrate again the first exact definition of symptotic formula (2.6)

$$\gamma^{-1} \sim f^2 \left[\frac{1}{n(n+1)} - \frac{s}{fn^2(n+1)^2}\right] . \qquad (2.19)$$

Another possible method for finding asymptotes is shown in Yaglom's article. It consists in finding the solution required for an infinite system of equations (2.17). This system is recurrent, therefore, the solutions always exist, at all the values of parameters f and . However, what we need is not just any b_n, but only those series at which the series (2.15) converge, i.e. b_n should tend to zero at $n \to \infty$ so, that $\sum' \tilde{b}_n^2 < \infty$. This condition should be obtained for selecting the eigen value for all the parameters. For asymptotic appraisal it may be assumed, that all $\tilde{b}_n$, for certain value of n, are equal to zero. Then instead of the infinite system of equations we get the finite system.

System (2.17) binds all factors through one. Therefore, it expands into two. For one of these $n = s, s + 2, \ldots,$ and for the other $n = s + 1, s + 3, \ldots$. The solutions correspondingly expand into those, for which Ψ is even, and odd, and into those, for which Ψ is odd, and π even.

In order that the system would have non-trivial solutions, its determinant must be equal to zero. For the first of the systems

$$\begin{vmatrix} \tilde{M}_s & \tilde{L}_s & 0 & 0 & \cdots & & \\ \tilde{L}_s & \tilde{M}_{s+2} & \tilde{L}_{s+2} & 0 & \cdots & & \\ 0 & \tilde{L}_{s+2} & \tilde{M}_{s+4} & \tilde{L}_{s+4} & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & & \cdots\ 0 & \tilde{L}_{N-2} & \tilde{M}_N \end{vmatrix} = 0, \qquad (2.20a)$$

and for the second

$$\begin{vmatrix} \tilde{M}_{s+1} & \tilde{L}_{s+1} & 0 & 0 & \cdots & & \\ \tilde{L}_{s+1} & \tilde{M}_{s+3} & \tilde{L}_{s+3} & 0 & \cdots & & \\ 0 & \tilde{L}_{s+3} & \tilde{M}_{s+5} & L_{s+5} & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & & \cdots\ 0 & \tilde{L}_{N-1} & \tilde{M}_{N+1} \end{vmatrix} = 0, \qquad (2.20b)$$

At $\gamma^{-1} \to \infty$ the diagonal elements approach to constants, and the non-diagonal elements have order $O(\gamma)$. In the expansion of determinant all products, except the product of diagonal elements, have order of at least $O(\gamma^2)$. Therefore, as the first approximation it is possible to assume determinant as equal to product of diagonal elements. In this case equation (2.20) is denoted simply as

$$\tilde{M}_s \cdot \tilde{M}_{s+2} \cdot \tilde{M}_{s+4} \cdots, \quad \tilde{M}_N = 0,$$

$$\tilde{M}_{s+1} \cdot \tilde{M}_{s+3} \cdot \tilde{M}_{s+5} \cdots \quad \tilde{M}_{N+1} = 0.$$

Hence it follows, that for every $n \geqslant s$ there is a solution at $\tilde{M}_n = 0$

$$\frac{s}{f} = n(n+1) + \frac{(n-1)^2(n+1)^2(n-s)(n+s)}{(2n-1)(2n+1)\left[(n-1)^2 n^2 \gamma^{-1} + sf - (n-1)\,nf^2\right]} +$$

$$+ \frac{n^2(n+2)^2(n-s+1)(n+s+1)}{(2n+1)(2n+3)\left[(n+1)^2(n+2)^2\gamma^{-1} + sf - (n+1)(n+2)\,f^2\right]}$$

The roughest approximation is simply $\frac{s}{f} = n(n+1)$. In the right-hand portion of the above formula f could be substituted by this rough approximation, and this will change the whole formula only to terms of further lower order. We will get

$$\frac{s}{f} = n(n+1) + \frac{(n-1)^2(n+1)^2(n-s)(n+s)}{(2n-1)(2n+1)\left[(n-1)^2n^2\gamma^{-1} + 2s^2n^{-1}(n+1)^2\right]} +$$

$$+ \frac{n^2(n+2)^2(n-s+1)(n+s+1)}{(2n+1)(2n+3)\left[(n+1)^2(n+2)^2\gamma^{-1} - 2s^2n^{-2}(n+1)^{-1}\right]} . \qquad (2.21)$$

This approximate formula was obtained by Hough. Further it will be seen, that it gives a good approximation in a wide range of γ^{-1} values, and only for the very low values of this parameter should be replaced by another formula, which will be carried out in the next chapter. It should be mentioned, that formula (2.18) could be obtained from (2.21) by expanding into series in order of γ . Formulas (2.21) and (2.18) are very similar in accuracy, and only for some low values of n and s the simpler formula (2.18) gives appreciably worse results.

Hough's formula (2.21) pertains to oscillations of second type. Now ina similar way, but with the help of (2.14) we shall deduce a formula for oscillations of the first type. Repeating exactly the above reasoning, we come to conclusion, that there is a need to find such values of parameters γ^{-1} and f, at which the following determinant will be equal to zero

$$\begin{vmatrix} M_s - \gamma^{-1} & L_s & 0 & 0 & \cdots & \\ L_s & M_{s+2} - \gamma^{-1} & L_{s+2} & 0 & \cdots & \\ 0 & L_{s+2} & M_{s+4} - \gamma^{-1} & L_{s+4} & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & & \cdots & L_{N-2} \; M_N - \gamma^{-1} \end{vmatrix} = 0. \tag{2.22a}$$

or determinant

$$\begin{vmatrix} M_{s+1} - \gamma^{-1} & L_{s+1} & 0 & 0 & \cdots & \\ L_{s+1} & M_{s+3} - \gamma^{-1} & L_{s+3} & 0 & \cdots & \\ 0 & L_{s+3} & M_{s+5} - \gamma^{-1} & L_{s+5} & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & & \cdots & L_{N-1} \; M_{N+1} - \gamma^{-1} \end{vmatrix} = 0. \tag{2.22b}$$

At $f \longrightarrow \infty$ the diagonal elements increase much faster than the non-diagonal / the first are of order $O(f^2)$, the second $O(1)$. Therefore, the approximate formulas are obtained by equating diagonal elements to zero

$$\gamma^{-1} = \frac{f^2 - 1}{\left(-\frac{s}{f} + n\right)\left(\frac{s}{f} - n - 1\right)} +$$

$$+ \frac{(n-s)(n+s)\left(\frac{s}{f}-n+1\right)}{(2n-1)(2n+1)\left(\frac{s}{f}+n\right)\left[\frac{s}{f}-n\,(n-1)\right]} +$$

$$+ \frac{(n-s+1)(n+s+1)\left(\frac{s}{f}+n+2\right)}{(2n+1)(2n+3)\left(\frac{s}{f}-n-1\right)\left[\frac{s}{f}-(n+1)(n+2)\right]} \,. \quad (2.23)$$

Formula (2.19) could be obtained from here by expansion into series according to the orders of $1/f$.

Formulas (2.21) and (2.23) represent oscillations of the second and first type at sufficiently high γ^{-1} values. However, on one hand, many practical problems require calculation of characteristic curves in the zone, where these approximate formulas are known to be useless. On the other hand, having only these formulas, it is impossible to judge, how far they are correct regarding the behavior of the characteristic curves of Laplace's equation as a whole. With this object these curves were calculated on electronic computer. The simplest method of calculation is the solution of characteristic equations (2.22). The convenience here is that one of the parameters, namely γ^{-1} enters in this equation in a " classical " way. Thus, by imparting partial values to parameter f, we get the usual problem on calculation of eigen or fundamental values of Jacobi type matrix. The differential equations (2.17) did not have this advantage. Both the parameters enter into these equations in the same inconvenient way.

Let us do some summing up. By raising the orders the Laplace's tidal equation was reduced to the two different forms, convenient for expansion into series according to Legendre polynomials. The form (2.9) permits to find a good expression for approximate values of f of the second type at high γ^{-1}, more exact, than the one resulting from the first form of equations. On the other hand, form (2.8) provides convenient expression for asymptotes of the first type at high γ^{-1}. The form (2.8) results in, what is, apparently, specially important, convenient method of numerical determination of eigen values of the parameters, in the next chapter it will be clear, that equations (2.8) make it possible to find the asymptotes at low values of γ^{-1}.

5. Results of characteristic curves.

Thus, it is possible to find characteristic curves of the Laplace's equation of the theory of tides by resolving characteristic equations (2.22a) and (2.22b). In practical calculations the determinants were of the twentieth order. Simultaneously with γ^{-1} eigen values the determination is also of characteristic vectors, i.e. sequence of a_n values - expansion factors of function Ψ from the set Legendre polynomials. If the eigne values was obtained from the system (2.22a), the expansion is according to functions $P^s{}_s$, $P^s{}_{s+2}$, $P^s{}_{s+4}$ ...., but if from (2.22b), then according functions $P^s{}_{s+1}$, $P^s{}_{s+3}$, $P^s{}_{s+5}$ ....,

In the first case Ψ is an even function μ, and ζ is odd, whereas in the second case - inversely. In the first case we name resolutions as even type resolutions, in the second case - odd type resolutions. Using the asymptotic formulas (2.21) and (2.23), it should be kept in view, that for resolving even type we should take in formula (2.21) $n = s + 1, s + 3, ...,$ and in formula (2.23) $n = s, s + 2, s + 4,$ and for odd type resolutions - inversely.

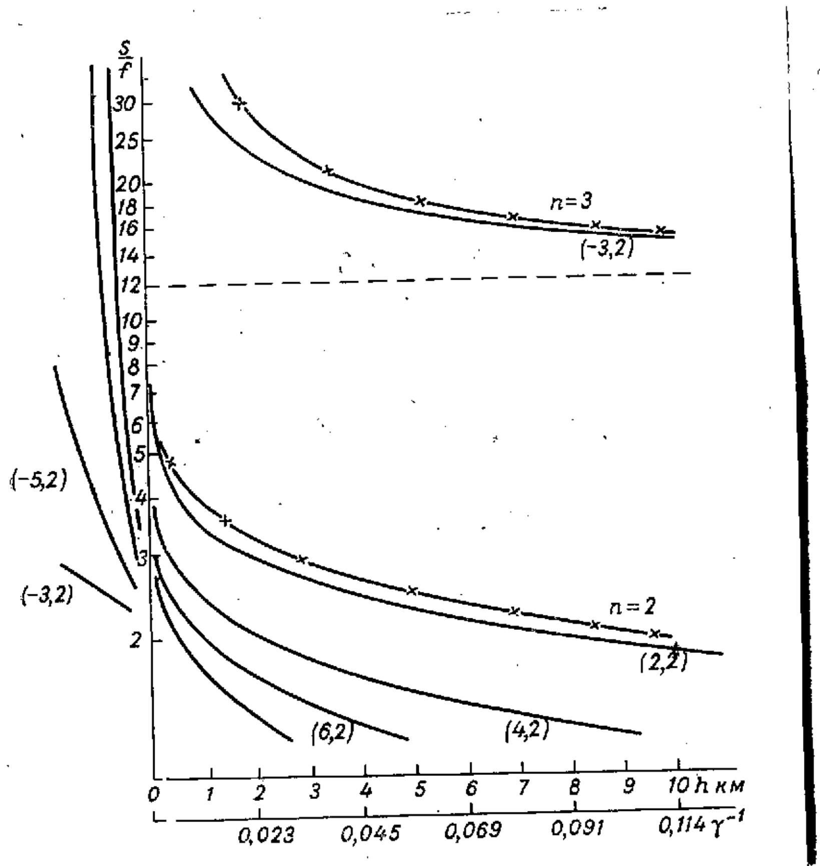

Fig. 2.2. Calculated characteristic curves.

Fig.2.2 shows results of calculations on electronic computer of even type curves at s = 2. On the right, at high γ^{-1} values, the curves are very well approximated by those marked with crosses, calculated from formulas (2.21) and (2.19) (the corresponding n values are written next to curves). In fig. 2.2 is seen a bunch of curves of the first type, approaching at $\gamma^{-1} \to \infty$ the axis of abscissae. At $\gamma^{-1} \longrightarrow 0$ they approach the axis of ordinates and are being approached on the right by the second type curves, each moving away from its asymptote $s/f = n(n+1)$. According to the proved, the curves cannot intersect. The meaning of figures in brackets will be explained in para 6.

The Fig. shows, that characteristic curves are present also to the left of the ordinates axis, i.e. at negative γ^{-1}. According to the proved in para 2, the negative eigen values of γ^{-1} could be only at $f \leqq 1$, i.e. $s/f \geqslant 2$. The Fig. shows, that immediately above the point $s/f = 2$ appears a bunch of characteristic curves, moving away to the left and upward. While rising they one by one separate from the bunch and approach from below the horizontal asymptotes - the same $s/f = n(n+1)$ asymptotes, as with positive γ^{-1} for the second type curves. The cal-

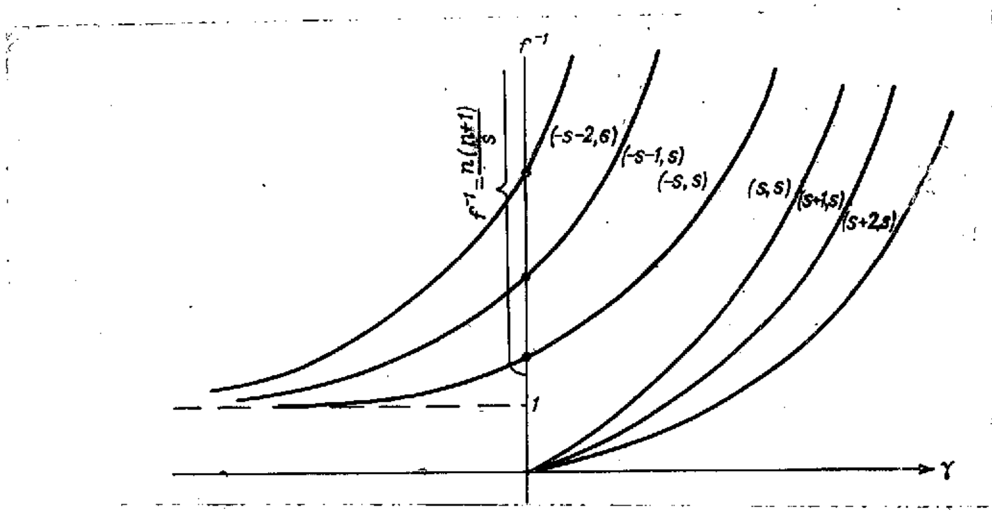

Fig. 2.3. Characteristic curves in coordinates f^{-1}, .

culations lose their accuracy at very low γ^{-1}, i.e. within the narrow band along the axis of ordinates. Therefore, the calculations do not make it possible to follow, how the characteristic curves in negative half-plane enter into axis of ordinates. Two assumptions are possible here: either they **all** gather at one point of the axis - point $s/f = 2$, or each of them arrives at its own point. In the next chapter it will be shown, that it is the first alternative that takes place.

In Fig. 2.3 a graph has been plotted of the same curves, as in Fig. 2.1 and 2.2, but in coordinates (f^{-1}, γ) (without adhering to scale). From this figure it is possible to discern, that it is quite natural to assume, that curves at negative γ and the curves of second type at positive γ are the same curves, continuously transient from the left half-plane into right. Nevertheless, coordinates (f^{-1}, γ^{-1}) will hence be used as before.

One of the sections of the first type top curve (n = 2) has many times attracted the attention of various investigatiors; calculation results are available in literature of this section's characteristic curve, for instance in Wilkes book (1949). We are speaking here of the environs of $f = 1$, i.e. $s/f = 2$. The frequency of oscillations here is $\sigma = 2\omega$, i.e. the period is half a day. The fundamental functions are asymptotically similar to $\sin(\sigma t + 2\varphi + \alpha_0)\ P_2^2\ (\mu)$, which corresponds to the following tidal structure: four junction meridians moving at rotation speed of the Earth, and the resolution does not convert into zero anywhere. These resolutions play the main role in the theory of tides. The value $s/f = 2$ corresponds, according to our calculations to $\gamma^{-1} = 0.0899$, which in conversion to dynamically equivalent depth h is 7.86 km. On the axis of abscissae are plotted also h values. Fundamental functions have the following appearance

$$\psi = a_2' P_2^2 + a_4' P_4^2 + a_6' P_6^2 + \ldots\ldots$$

For $s/f = 2$ our calculations give

$$\psi = P_2^2 - 0{,}08700P_4^2 + 0{,}004433P_6^2 - 0{,}000048P_8^2 + \ldots \qquad (2.24)$$

The results are similar to those, obtained by Peckeris (1937) see also Wilkes, (1949) .

The theory of tides is dealt with in detail in Wilkes' book, and also in Siebert's review (1961). We point out for comparison, that for the next, second from the top curve of first type, i.e. at $n = 4$ the fundamental function at $s/f = 2$ is such :

$$\psi = 0{,}576\ P_2^2 + P_4^2 - 0{,}434\ P_6^2 + 0{,}024\ P_8^2 + \ldots.$$

Here the predominant is the second component. Fig. 2.4 shows similar curves, but for negative f values, which, as we know, means wave propagation west to east. These curves are distinct by the absence here of the second type curves.

It is most instructive to plot the same curves, as in Fig. 2.1, but in logrithmic scale, which clearly shows every exponential relation. Fig. 2.5c - shows this type of curves. Fig. 2.5 a shows the case of $s = 1$, Fig. 2.5b - $s = 2$, Fig. 2.5c - the case of $s = 3$. Each of these figures contains curves of even, as well as odd type. In each figure there are one or two curves of the first type ($n = s,\ s + 1$) and three bottom curves of the second type. By short dotted lines are drawn asymptotic curves (2.21) and (2123). Here again it is clearly shown, that these asymptotic formulas quite aatisfactorily depict characteristic curves at $\gamma^{-1} > 0.05 - 0.1$. In the figures it is clear, that with reduction of $\gamma^{=1}$ almost at once begins to act another asymptotic, as the curves cnnvert into straight lines. Long dashes show the straight lines, which are asymptotically approached by our characteristic curves. We can see, moreover, that this new asymptotic at low γ^{-1} also divide all curves into two bunches. The top bunch corresponds to exponential relation $f\gamma^{1/2} = const$, and the bottom - to relation $f^2\gamma^{1/2} = const$. The number p in the figure will be explained in the next chapter; this is a number in the corresponding formula of asymptotics.

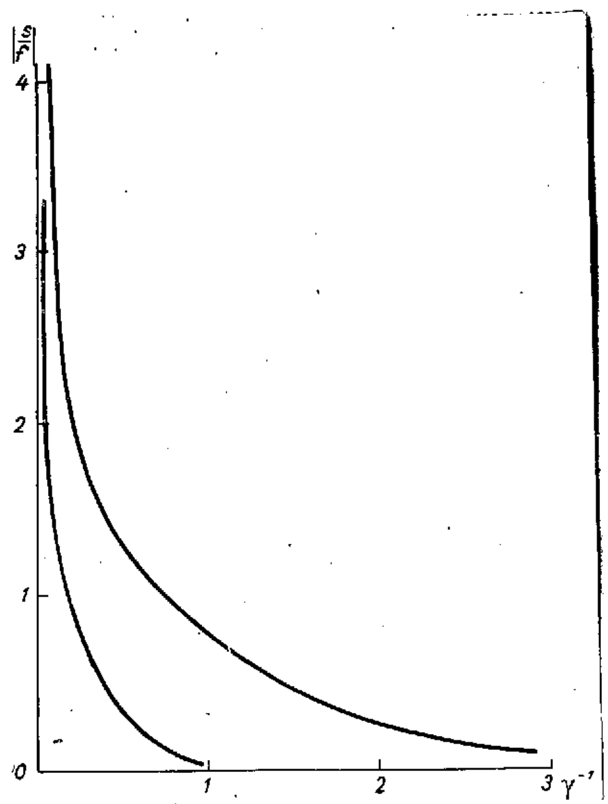

Fig. 2.4. Characteristic curves for s = 2, f 0.

As will be seen further, the so called Rossbi waves, which are of high significance in meteorology, correspondsfor the case of the earth atmosphere to γ^{-1} values of 0.1 order. Therefore, for their depiction the asymptotics, obtained by us previously, are quite sufficient. However, many other questions of the theory of oscillations are bound with shallow depths h (or γ^{-1}) and the behavior of characteristic curves at these low values becomes of considerable interest. The fact, just remarked, of the presence of certain exponential asymptotics at low γ^{-1}, discovered during calculations of characteristic curves on electronic computer, made it necessary to investigate this question in a theoretical way. The obtaining of this type of asymptotics will be dealt with in the next chapter. It should be mentioned, that finding of asymptotics at low γ^{-1} requires finer means, than the finding of asymptotics at high γ^{-1}. If in the finding of asymptotics at high γ^{-1} it was possible to omit the low terms of equations, here we are face to face with the problem of low parameter at senior derivative.

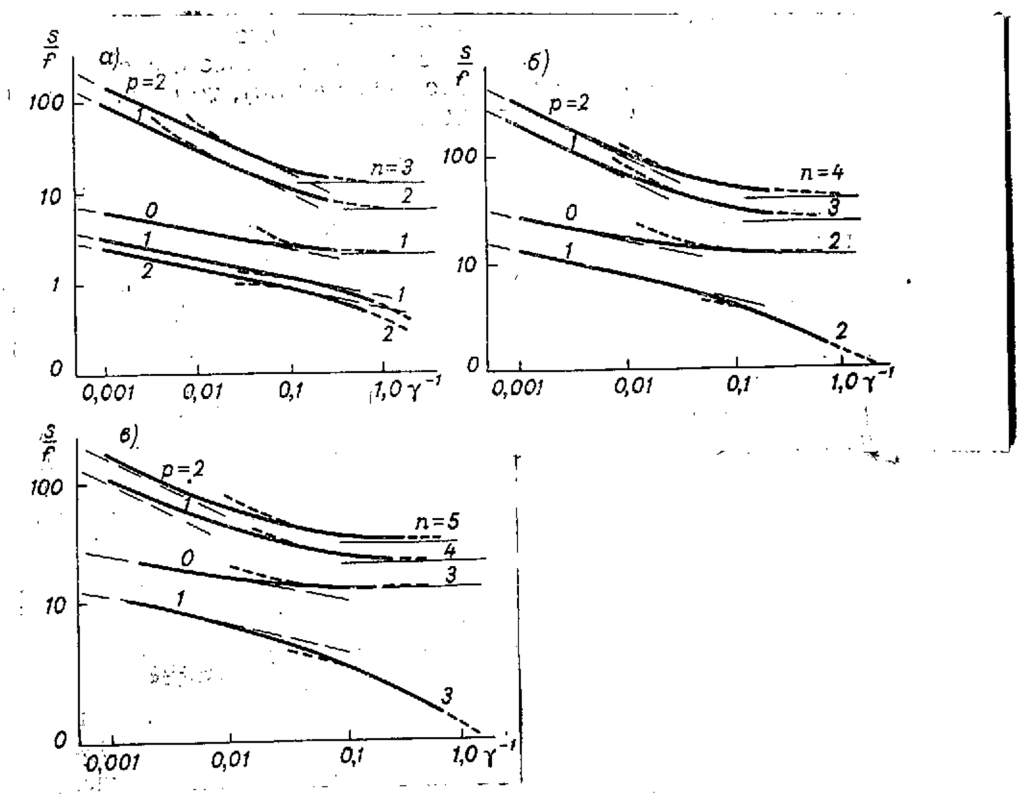

Fig. 2.5. Characteristic curves in logarithmic coordinates. a) s = 1, b) s = 2, c) s = 3.

We have already mentioned the double ultimate transition $\omega \to 0$), $a \longrightarrow \infty$, i.e. transition to a model of flat non-rotatory Earth. What happens in this case with characteristic curves? If is directed towards zero, the system (2.3) will be

$$(1-\mu^2)\frac{d\psi}{d\mu} = \eta \quad (\eta = f^2 \zeta),$$

$$(1-\mu^2)\frac{d\eta}{d\mu} = \left[s^2 - (1-\mu^2)\, a^2 \frac{\gamma^2}{gh} \right]\psi \quad ,$$

whence

$$(1-\mu^2)\psi - 2\mu\psi - \frac{s^2}{1-\mu 2}\psi + \frac{a^2\gamma}{gh}\psi = 0,$$

i.e. ψ is Legendre function $P^s{}_n$, $a^2 \sigma^2 / gh = n(n+1)$. The whole curve coincides with asymptotics of the first type. Horezontal asymptotes have withdrawn into infinity. If now $a \longrightarrow \infty$, the curves concentrate limitlessly, filling the whole plane, the spectrum converts into continuous. If it is assumed, that jointly with a increases also n, so that $n/a \longrightarrow k$ (k-wave number), the characteristic curve for the given k will be

$$\sigma = \sqrt{gh}\; k,$$

i.e. phase velocity of wave is equal to $\sqrt{gh}$, which is what was pointed out in preceding chapter. For very long waves, commensurable with dimensions of the Earth, the concept of phase velocity loses its meaning.

6. Wave type and the number of resolution points.

The eigenvalues, disposed on one and the same curve, belong, as it is said, to one and the same mode or wave type. What characterizes this wave type? As long as one of the asymptotes is active at high γ^{-1}, i.e. at the base of each curve, the fundamental function ψ is similar to spherical function (for the curves of the first type) or to linear combination of two such functions (for the curves of second type). In this case the number of the curve characterizes the number of points of the fundamental function. Now we'll introduce denotations for individual modes. Each curve will be marked by a double number (n, s); s - is familiar to us azimuthal wave number, number n for curves of the first type will have the values $s, s + 1, s + 2, \ldots$, i.e. this is the n number, which participates in asymptotics $\psi \sim P^s_n(\mu)$ for the given mode. For the curves of second type we shall provisionally write negative number $n < 0$, in order to distinguish them from the curves of first type. In this case n will take on the values - $- s, - s - 1, - s - 2, \ldots$. Thus, for mode (n, s), $n < 0$ the asymptotics at $\gamma^{-1} \to \infty$ will be : $\xi \sim P^s_{|n|}(\mu)$. Functions ψ , ξ , pertaining to mode (n, s), we denote through $\psi^s_{n,f}(\mu)$, $\xi^s_{n,f}(\mu)$.

As mentioned earlier, the curves in half-plane of negative γ^{-1} would be quite natural to consider as continuation of the second type curves in positive half-plane (at $f > 0$, of course). From one half-plane they pass into another, breaking along the asymptotes $s/f = n(n + 1)$. Therefore, for the entire mode as a whole, both in the region of positive and negative γ^{-1} we retain the same denotation

(n, s), $\psi^s_{n,f}$, $\xi^s_{n,f}$. At $f < s/n(n+1)$ the curve lies in the half-plane $\gamma^{-1} > 0$, and at $f > s/n(n+1)$ - in half-plane $\gamma^{-1} < 0$. At $f < 0$ the whole curve lies entirely in the region $\gamma^{-1} < 0$.

The question regarding the number of zeros of the fundamental function is highly important, and a lot here is not clear to the end. In the action region of asymptotics at high γ^{-1} the number of zeros could be found. Thus, for asymptotics of the first type $n > 0$, the number of zeros in function $\psi^s_{n,f}(\mu)$ is equal to n - s, and for function $\xi^s_{n,f}(\mu)$ one unit higher, n - s + 1 (we do not count the zeros of these functions at the end of interval). For asymptotics of the second type, $n < 0$, the number of zeros in function $\xi^s_{n,f}(\mu)$ is equal to $|n| - s$, and for function $\psi^s_{n,f}(\mu)$ it is expressed, as will be shown in chapter 3, in a more composite way: it is equal to $|n| - s + 1$, when $|n| - s$ is even, and $|n| - s - 1$, when $|n| - s$ is odd.

Can it be said, that the number of points of the fundamental function is retained along the whole mode? The usual proof of retaining the number of zeros in the fundamental function, based on the theorem of si singleness for differential equation of the second order, is complicated by the existance of special points, firstly, at the ends of the segment, secondly, in critical latitudes. The usual expansion into series according to orders of $(\mu \pm 1)$ shows, that functions ψ, ξ have at the ends of segment $[-1, 1]$ have a special feature of the type $(1-\mu^2)^{s/2}$ But at f = + 1 the zero order of function rises by a unit, becoming equal to $(1-\mu^2)^{(s/2)+1}$. This indicates, that with positive f, varying and passing through f = 1, the number of points changes by two. It is precisely with decreasing f that two points get added; it is not possible for the two points to get lost, since from the asymptotics at high f it can be seen, that the extreme, nearest to ends of segment $[-1, 1]$ are the ξ zeros, and not ψ. Therefore, the ψ radicals cannot leave the segment. As regards function ξ , the zero order of this function at the ends of segment is always the same, i.e. no zero can either enter, nor leave through the ends of segment.

Another possibility of the changing number of points is connected with existence of critical latitudes. At points $\mu = \pm f$ exist resolutions $\psi(\mu)$, which convert into zero jointly with their first derivatives. Due to this with continuous variation of parameters the number of points in function ψ could change at once by two in each critical point, i.e. by four. In exactly the same way zero number of function ξ can change at once by four at points on axis μ , in which conversion into zero occurs of $(s^2 / f^2) - (1-\mu^2)\gamma$(these points could be denoted as critical points of the second order). Beforehand the possibility cannot be eliminated of such jump-like variation of the number of points in fundamental functions. It is only possible to indicate those zones of parameter values, where it cannot take place. This is, firstly, the zone of negative γ , secondly, zone $s^2 > f^2_{\gamma}$ and, thirdly, zone $|f| > 1$. In each of these cases there are no critical points either of the first, or second order. It is obvious, that if there are no critical points of the first order ($|f| > 1$), the number of points in function ψ cannot change, and if there are no points of the second order ($\gamma < 0$ or $s^2 > f^2\gamma$), there can be no change in the number of points in function . But it happens, that in the first case the number of nodes in function ξ is also invariable, and in the second case of function ψ. Indeed, satisfies equation, which is obtained from system (2.3) by exclusion of ψ

$$\left[\frac{d}{d\mu} - \frac{s}{f(1-\mu^2)}\right] \frac{1-\mu^2}{\frac{s^2}{f^2} - (1-\mu^2)\gamma}\left[\frac{d}{d\mu} + \frac{s\mu}{f(1-\mu^2)}\right]\xi =$$

$$= \frac{f^2 - \mu^2}{1-\mu^2}\,\xi .$$

Assuming, that at the critical point of second order $\mu = \mu^*$, where $s^2/f^2 - (1-\mu^2)\gamma$ converts into zero, ξ and ξ' are equal to zero, in which case $|f| > 1$. We multiply the above equation by ξ and integrate from μ^* to 1. After partial integration we'll have

$$-\int_{\mu^*}^{1} \frac{1-\mu^2}{\frac{s^2}{f^2} - (1-\mu^2)\gamma}\left[\frac{d\xi}{d\mu} + \frac{s\mu\xi}{f(1-\mu^2)}\right]^2 d\mu =$$

$$= \int_{\mu^*}^{1} \frac{(f^2-\mu^2)\,\xi^2\,d\mu}{1-\mu^2} .$$

Here the left portion is negative, whereas the right positive, which cannot be. In the same way at $\gamma < 0$ or $s^2/f^2 > \gamma$, if at the critical point of the first order $\mu = f$ function ψ and its derivative ψ convert into zero, then by multiplying equation (2.2) by ψ and integrating from $-f$ to f, we'll have

$$-\int_{-f}^{f} \frac{1 - \mu^2}{f^2 - \mu^2} \left[\frac{d\psi}{d\mu} - \frac{s\mu\psi}{f(1 - \mu^2)} \right]^2 d\mu =$$

$$= \int_{-f}^{f} \left[\frac{s^2}{f^2(1 - \mu^2)} - \gamma \right] \psi^2 \, d\mu \; .$$

And here too the left and right portions are of different signs. Even this incomplete information regarding the number of zeros will be sufficient to confirm substantially in the next chapter disposition of characteristic curves.

7. A note on expansion according to Hough's functions.

It is quite common in meteorology to use expansion of various types of fields according to Legendre functions. It would be more natural in many cases to expand in accordance with Hough's functions, specially if they are properly tabulated. The basis for this type of expansion is the theorem regarding completeness of the system of Hough's functions. To be more exact, with all possible eigenvalues of γ is complete on the segment $[-1,1]$[1]. The proof of this theorem is not being given, since it is in no way different from the usual proofs of completeness of fundamental functions system for the problem of Schturm-Liuville. For equation (2.2) the plotting is of Green's function, i.e. the inverse integral operator. The Green's function is continuous, since it is plotted from continuous resolutions. For integral operator the usual theorems of the completeness of fundamental functions hold true.

---

[1] With f values, at which there are breaks of characteristic curves, i.e. at $s/f = n(n + 1)$, the attention for completeness should be paid also to $\gamma = 0$, corresponding to which is the fundamental functions $\xi = P^s_n(\mu)$.

The only difference from the usually encountered equations is the fact, that among the eigenvalues there could be an infinite amount of positive, as well as negative.

It was proved above, that negative values could exist only for oscillations with period of less than half a day, i.e. at $|f| > 1$. In the next chapter it will be shown, that at $|f| = 1$ there is virtually an immediate appearance of all the negative eigenvalues, i.e. all the characteristic curves, lying in the negative half-plane in Fig. 2.1 , proceed from one point $|f| = 1$ on axis f^{-1}. At first sight the situation appears extremely strange. Assuming, $|f|$ decreases from values higher than unit, to values lower than unit. Initially the eigenvalues were only positive, and the corresponding fundamental functions formed a complete system. With continuous variation of parameter $|f|$ these fundamental functions continuously vary and at a certain moment it suddenly becomes necessary to add at once an infinite lot of new functions in order to obtain a complete system of functions, i.e. the old system very abruptly ceases to be complete.

The explanation of this event will be obtained by analyzing the nature of fundamental functions at $|f|$ values, approximating unit. We demonstrate, that at $|f|$, a little less than unit, there is formation next to ends of segment $[-1, 1]$ of narrow zones, in which Hough's functions (at $\gamma > 0$) lose the oscillation nature, i.e. become of constant sign. With decreasing $|f|$ these zones widen out. At very low $|f|$ values the functions may oscillate in a very narrow zone around the equator. An appearance of the norrowest non-oscillation zone is enough for the system of functions to cease being complete; in fact it is quite apparent, that this system of functions cannot be used for expansion, for instance, of functions, which differ from zero only in the indicated non-oscillation zone. On the contrary, Hough's functions, corresponding to $\gamma < 0$, oscillate only in zones close to poles, i.e. precisely these are the best suited for expansion of functions, distinct from zero in these zones. Thus, there are virtually two additional systems of functions.

Let's check these assertions. Assuming there is some extreme point of function Ψ. Then at this point, as follows from the Laplace equation (2.2), will be

$$(1-\mu^2)\,\psi'' = \left[\frac{s}{f}\,\frac{f^2+\mu^2}{f^2-\mu^2} + \frac{s^2}{1-\mu^2} - \gamma\,(f^2 - \mu^2)\right]\psi \;.$$

If, $\gamma < 0$, the term in square brackets is positive at the segment $[-f, f]$. Therefore, ψ'' has the same sign, as ψ. At the extremum point the curve is directed with convexity to the axis of abscissae. Therefore there could only be one extremum point (at the origin of coordinates) and only one intersection with the axis (at the same place). Thus, at $\gamma < 0$ segment $[-f, f]$ is the segment of non-oscillation of resolution.

Assuming now, that $\gamma > 0$. Let's analyse, inversely, additional segment $[-1, -f]$ and $[f, 1]$. The two last terms in the square brackets of the above relation are here positive. True, the first term is negative. But if a narrower segment is taken, then with sufficiently high γ, i.e. for all Hough's functions, starting from any, the last term will be higher in absolute value and the whole formula will be positive. If it is taken into account, that the resolution should convert into zero at $\mu = \pm 1$, it is possible to come to conclusion, that there will be no extremums, and therefore, no zeros. In the next chapter will be given a more precise evaluation of the boundaries of non-oscillation zone.

How to expand an arbitrary function from Hough's functions? If tables are made up of these functions, their expansion would be no more difficult than from Legendre functions. But an approach could be made to similar expansion also from expansion by Legendre functions, using expansion of Hough's function from the set Legendre functions. In other words, there is a transition matrix from one orthogonal system to another. Since such a matrix is orthogonal, the inverse matrix will be transposed, i.e. obtained by replacement of lines by columns. For example we are giving Table 2.1 of expansion factors of Hough's functions from the set Legendre functions, calculated for a random case s = 3, f = 0,781. The Table gives γ^{-1} values, corresponding to Hough functions f = 0.781 and expansion factors of Hough's functions from the set Legendre functions or vice cersa - Legendre functions from Hough's functions. The matrix is orthogonal.

Expansion accordingto Hough'sfunctions for positive and negative has been used by Lindzen (1966,1967), Kato (1966) and Zhantuarov (1967) in the theory of thermal diurnal tides in the atmosphere.

TABLE : 2.1

Expansion of Hough's functions from the set Legendre functions for the case of s = 3, f = 0.781.

	γ^{-1}	$\bar{P}_3^3$	$\bar{P}_5^3$	$\bar{P}_7^3$	$\bar{P}_9^3$	$\bar{P}_{11}^3$	$\bar{P}_{13}^3$	$\bar{P}_{15}^3$
ψ_3^3	0.0288	0.9356	-0.3462	0.0693	-0.0083	0.0006	0.0001	0.0000
ψ_5^3	0.0083	-0.2539	-0.5364	0.7024	-0.3742	0.1173	-0.0251	0.0041
ψ_7^3	0.0037	0.1272	0.3391	-0.0907	-0.5123	0.6435	-0.3960	0.1583
ψ_{-4}^3	-0.0037	0.1363	0.4864	0.6467	0.4992	0.2589	0.0975	0.0279
ψ_9^3	0.0021	0.0795	0.2278	0.0221	-0.3613	0.0510	0.4700	-0.6070
ψ_{-6}^3	-0.0014	-0.0725	-0.2388	-0.2028	0.1251	0.4748	0.5859	0.4665
ψ_{11}^3	0.0013	-0.0553	-0.1635	-0.0430	0.2450	0.0928	-0.3583	0.0501

SUPPLEMENT TO CHAPTER 2.

Equivalence of various forms of Laplace equation of the theory of tides.

We demonstrate the previously formulated suggestion, that the system (2.8), which is the issue of system (2.3), has no extraneous eigenvalues of parameter s/f, except a whole number ones, at any γ. For convenience we introduce the following denotations

$$D_+ = (1-\mu^2)\frac{d}{d\mu} + \frac{s\mu}{f},$$

$$D_- = (1-\mu^2)\frac{d}{d\mu} - \frac{s\mu}{f}.$$

The left portions of equations (2.3), in which all terms are transferred to the left, are denoted by A and B:

$$A = \left[(1-\mu^2)\frac{d}{d\mu} - \frac{s\mu}{f}\right]\psi - (f^2-\mu^2)\xi,$$

$$B = \left[(1-\mu^2)\frac{d}{d\mu} + \frac{s\mu}{f}\right]\xi - \left[\frac{s^2}{f^2} - (1-\mu^2)\gamma\right]\psi.$$

In the same way also for equations (2.8) we take:

$$\tilde{A} = \left(L + \frac{s}{f}\right)\xi - \gamma\left[\left(\frac{s}{f}+2\right)\mu - (1-\mu^2)\frac{d}{d\mu}\right]\psi,$$

$$\tilde{B} = \left[L - \frac{s}{f} + \frac{s^2}{f^2} + (f^2-1)\gamma\right]\psi -$$

$$-\left[\left(\frac{s}{f}-2\right)\mu+(1-\mu^2)\frac{d}{d\mu}\right]\xi,$$

It is not difficult by direct estimate to be convinced, that

$$D_+A=\widetilde{B}(1-\mu^2)+(1-f^2)B,$$

$$D_-B=\widetilde{A}(1-\mu^2)-\frac{s^2}{f^2}A. \qquad (2.26)$$

These identities permit to clarify the question regarding equivalence of systems (2.3) and (2.8), i.e. of systems

$$\begin{cases} A\equiv 0 \\ B\equiv 0 \end{cases} \text{ and } \begin{cases} \widetilde{A}\equiv 0 \\ B\equiv 0. \end{cases}$$

It is obvious, that from $A\equiv B\equiv 0$ it follows, that $\widetilde{A}\equiv\widetilde{B}\equiv 0$, i.e. that the system (2.8) is the issue of system (2.3). Assuming now, inversely, $\widetilde{A}\equiv\widetilde{B}\equiv 0$. If in this case $B\equiv 0$, then also $A\equiv 0$, i.e. the system (2.3) is met and the resolution is not extraneous. Assuming $B\not\equiv 0$. Then from (2.26) we have :

$$D+A=(1-f^2)B,$$

$$D\_B=-\frac{s^2}{f^2}A.$$

Excluding A from here, we'll have

$$D+D\_B=-(1-f^2)\frac{s^2}{f^2}B,$$

or otherwise

$$\left[\frac{d}{d\mu} (1 - \mu^2) \frac{d}{d\mu} - \frac{s}{f} + \frac{s^2}{f^2} - \frac{s^2}{1 - \mu^2} \right] B = 0,$$

i.e. B satisfies Legendre equation. This may be in the case, if

$$\frac{s^2}{f^2} - \frac{s}{f} = n(n + 1), \ n \geqslant s,$$

i.e. $s/f = n + 1$ or $s/f = - n$. In this case $B = P_n^s$ and

$$A = - \frac{f^2}{s^2} D - P_n^s = C_1 \ P_{n+1}^s,$$

or $A = C_2 \ P_{n-1}^s$.

Thus, if (2.8) has resolutions, not meeting (2.3), this could only be at

$$\frac{s}{f} = \pm \ n \ (\ n > s \qquad n \leqslant -s \ ).$$

Other extraneous resolutions there cannot be. Actually for us only this is important, but for completeness the question should also have been clarified, as to whether the values $\frac{s}{f} = \pm n$ are actually eigenvalues, i.e. whether the system (2.8) always has a resolution at the whole s/f and any γ^{-1}. We wont try to prove it very precisely, and will restrict ourselves to the following, on the whole, quite convincing reasoning. Assuming, for instance, $s/f = n + 1$. Then we have to confirm the existence of the system's resolution

$$A \equiv \left[(1-\mu^2)\frac{d}{d\mu} - \frac{s\mu}{f} \right]\psi - (f^2-\mu^2)\,\xi = C_1 P^s_{n+1},$$

$$B \equiv \left[(1-\mu^2)\frac{d}{d\mu} + \frac{s\mu}{f} \right]\xi - \left[\frac{s^2}{f^2} - (1-\mu^2)\gamma \right]\psi = P^s_n.$$

The corresponding homogeneous system $A \equiv B \equiv 0$, as was assumed, does not have non-trivial resolutions, therefore, according to the usual alternative, they should be in the non-uniform system. These are the extraneous resolutions. The only weak point in this reasoning is that the correctness has not been proved of this alternative in the given situation.

From the system of equations (2.8) it is possible to obtain one more form of Laplace equations. We substitute in the right portions of this system the terms

$$\gamma\left[\frac{s\mu}{f} - (1-\mu^2)\frac{d}{d\mu} \right]\psi \quad \text{and}$$

$$\left[\frac{s\mu}{f} + (1-\mu^2)\frac{d}{d\mu} \right]\xi,$$

using equations (2.3). We get system

$$(L + \frac{s}{f})\,\xi + \gamma\,(f^2-\mu^2)\,\xi - 2\mu\gamma\psi = 0,$$

$$(L - \frac{s}{f})\,\psi + \gamma\,(f^2-\mu^2)\,\psi + 2\mu\,\xi = 0. \qquad (2.27)$$

This system cannot be totally equivalent to system (2.3) due to the following. If f at a given γ is eigenvalue of the system (2.27), then -f is also eigen value. (Actually, by denoting $\xi_1 = -\gamma\psi, \psi_1 = \xi$, we will find, that ξ_1, ψ_1 satisfy system (2.27) at $f_1 = -f$[1]). At the same time system (2.3) does not possess this property. System (2.27) was obtained as an issue of system (2.3), therefore, any eigenvalue of system (2.3) is an eigenvalue of system (2.27), but not vice versa. We will demonstrate, that if f is an eigen value of system (2.27), then either f, or -f is an eigenvalue of system (2.3). We denote the left portions of equations (2.27) respectively through $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$. Then, as it is not difficult to check,

$$D_+A = \tilde{\tilde{B}}\,(1-\mu^2) + (\mu^2 - f^2)\,B,$$

$$D_-B = \tilde{\tilde{A}}\,(1-\mu^2) - \left[\frac{s^2}{f^2} - (1-\mu^2)\gamma\right] A. \qquad (2.28)$$

Assuming now that ξ, ψ at a given f satisfy system (2.27), i.e. $\tilde{\tilde{A}} \equiv \tilde{\tilde{B}} \equiv 0$. Then either $A \equiv B \equiv 0$, i.e. the system (2.3) is also satisfied, and f is the eigen value of this system, or A and B are the non-trivial resolutions of system (2.28), which in this case converts into a system

$$\left[(1-\mu^2)\frac{d}{d\mu} + \frac{s\mu}{f}\right] A = (f^2 - \mu^2)\,B,$$

$$\left[(1-\mu^2)\frac{d}{d} - \frac{s\mu}{f}\right] B = \left[\frac{s^2}{f^2} - (1-\mu^2)\gamma\right] A,$$

---

[1] It should be mentioned, that resolutions, corresponding to f and -f, have different evennes; if for one of them ψ even, and ξ odd, then for the other it is vice-versa.

which coincides with (2.3), in which ψ and ξ are substituted by A and B, and f by -f, i.e. the system (2.3) in this case has eigen value -f.

Finally, let's turn to system (2.9), used in Yaglom's work (1953). We demonstrate that it is an issue of system (2.3), where Ψ, Φ and π are determined as described in para 3, by formulas (2.9c), (2.10) and (2.11). The left portion of equation (2.9c) we convert, using formulas (2.9c),(2.10) and (2.11)

$$\mathrm{if}L\Psi + \mathrm{is}\Psi + \mu L\Phi + (1-\mu^2)\frac{d\Phi}{d\mu} =$$

$$= \mathrm{if}\left(L\Psi + \xi + \frac{\mu}{\mathrm{if}}L\Phi\right) = \mathrm{if}(L\Psi + \xi - \gamma\mu\pi) =$$

$$= \frac{\mathrm{if}^2 L}{s}\left[\xi - \frac{(1-\mu^2)}{\mathrm{if}}\frac{d\Phi}{d\mu}\right] + \mathrm{if}\xi - \mathrm{if}\gamma\mu\pi .$$

If we use relation

$$L(1-\mu^2)\frac{d}{d\mu} = (1-\mu^2)\frac{d}{d\mu}L - 2\mu L,$$

then the preceding formula will get converted into

$$\frac{\mathrm{if}^2}{s}\left\{L\xi - \frac{(1-\mu^2)}{\mathrm{if}}\frac{d}{d\mu}(L\Phi) + \frac{2\mu}{\mathrm{if}}L\Phi + \frac{s}{f}\xi - \frac{\gamma\mu s}{f}\pi\right\} =$$

$$= \frac{\mathrm{if}^2}{s}\left\{L\xi + \gamma(1-\mu^2)\frac{d\pi}{d\mu} - 2\gamma\mu\pi + \frac{s}{f}\xi - \frac{\gamma\mu s}{f}\pi\right\} =$$

$$= \frac{if^2}{s}\left\{ (L + \frac{s}{f})\xi - \gamma\left[(\frac{s}{f} + 2)\mu - (1 - \mu^2)\frac{d}{d\mu}\right]\psi \right\} .$$

But this formula is equal to zero by force of equation (2.8), which is an issue of equation (2.3). Therefore, equation (2.9a) has been satisfied.

In order to obtain the second equation (2.9), we apply to both portions of (2.11) operator $(1 - \mu^2) \frac{d}{d\mu}$:

$$(1 - \mu^2)\frac{d\xi}{d\mu} = \frac{s}{f}(1 - \mu^2)\frac{d\Psi}{d\mu} + \frac{1}{if}(1 - \mu^2)\frac{d}{d\mu}(1 - \mu^2)\frac{d\Phi}{d\mu},$$

or

$$(1 - \mu^2)\frac{d\Psi}{d\mu} = \frac{s}{f}(1 - \mu^2)\frac{d\xi}{d\mu} - \frac{1}{is}(1 - \mu^2)\left[L\Phi + \frac{s^2}{1 - \mu^2}\Phi\right],$$

or

$$(1 - \mu^2)\frac{d\Psi}{d\mu} = \frac{f}{s}(1 - \mu^2)\frac{d\xi}{d\mu} + \frac{\gamma f}{s}(1 - \mu^2)\pi + is\Phi .$$

From (2.3)

$$\gamma(1 - \mu^2)\pi = \gamma(1 - \mu^2)\psi = \frac{s^2}{f^2}\psi - (1 - \mu^2)\frac{d\xi}{d\mu} - \frac{s\mu}{f}\xi . \tag{2.29}$$

Therefore,

$$(1 - \mu^2)\frac{d\Psi}{d\mu} = \frac{f}{s}(1 - \mu^2)\frac{d\xi}{d\mu} + \frac{s}{f}\psi - (1 - \mu^2)\frac{d\xi}{d\mu} -$$

$$-\mu\xi + is\Phi = \frac{s}{f}\psi - \mu\xi + is\Phi .$$

Substituting this term into the left portion of (2.9b), with the use of proved formula $L\Psi + \xi - \gamma\mu\pi = 0$. We get

$$f^2\gamma\pi + is\Phi + \mu\,(\xi - \gamma\mu\pi) - \frac{s}{f}\psi + \mu\xi - is\Phi + L\pi =$$

$$= f^2\gamma\pi + 2\mu\xi - \gamma\mu^2\pi - \frac{s}{f}\psi + L\pi \;.$$

Using once more (2.29) ;

$$f^2\gamma\psi + L\psi + 2\mu\xi - (1-\mu^2)\frac{d\xi}{d\mu} - \frac{s\mu}{f}\xi + \frac{s^2}{f^2}\psi - \gamma\psi - \frac{s}{f}\psi =$$

$$= \left\{ \left[L + \frac{s^2}{f^2} - \frac{s}{f} + \gamma\,(f^2 - 1)\right]\psi - \left[(\frac{s}{f} - 2)\mu + \right.\right.$$

$$\left.\left. + (1 - \mu^2)\frac{d}{d\mu}\right]\xi \right\} .$$

The last formula is equal to zero due to the second of equations (2.8). Thus, the equation (2.9b) also happens to be fulfilled.

Chapter - 3

CHARACTERISTIC AYSMPTOTIC CURVES OF LAPLACE'S EQUATION OF THE THEORY OF TIDES FOR LOW DYNAMICALLY EQUIVALENT DEPTHS.

1. WORKING SYSTEM OF EQUATIONS :

At the end of the previous chapter a problem was set to investigate the behavior of the characteristic asymptotic/of curves laplace's equation of the theory of tides at $\gamma^{-1} \longrightarrow 0$ (or $h \rightarrow 0$). So far we only know that at $\gamma^{-1} \rightarrow 0$ along the curve $f \rightarrow \infty$. In this chapter formulas will be given for the first terms of asmptotes. This investigation is made in a separate chapter, since according to the applied mathematical device it is highly distinct from the device of the preceding chapter. The contents of this chapter are briefly published in the author's articles (1966, 1968), supplement to result H in Golitsin and Dikii article (1966).

Similar results were obtained by Longuet-Higgin (1968).

In the preceding chapter it was shown that the issue of system (2.3), which is properly called the tidal laplace's equation is a system of the fourth order (2.8). In deduction of asymptotes this system, or more exactly, the first equation of the system, will play the main role. We recall this equation.

$$(L + \frac{s}{f}) \xi = \gamma \left[(\frac{s}{f} + 2)\mu - (1 - \mu^2) \frac{d}{d\mu} \right] \psi , \qquad (3.1)$$

Where

$$L = \frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu} - \frac{s^2}{1-\mu^2}$$

The first simplification, which could be made in connection with the fact that the search is for asmptotes at $\gamma \to \infty$ and $f \to 0$, is to disregard two in comparison with increasing term s/f. The equation then will be

$$(L + \frac{s}{f})\xi = \gamma\left[\frac{s\mu}{f} - (1-\mu^2)\frac{d}{d\mu}\right]\psi \tag{3.2}$$

Let us remember now the first of equations (2.3)

$$\left[(1-\mu^2)\frac{d}{d\mu} - \frac{s\mu}{f}\right]\psi = (f^2-\mu^2)\xi \cdot \tag{3.3}$$

From (3.2) and (3.3) it follows

$$(L + \frac{s}{f})\xi = -\gamma(f^2-\mu^2)\xi \cdot$$

If we denote

$$E = f^2 + \frac{s}{f\gamma}, \tag{3.4}$$

We will get

$$-\frac{d}{d\mu}(1-\mu^2)\frac{d\xi}{d\mu} + \frac{s^2}{1-\mu^2}\xi + \gamma(\mu^2 - E)\xi = 0. \tag{3.5}$$

This is the equation that will now be the main one. We have to study variation of eigen values E at unlimited increase of parameter γ. This is a problem on asymptotic behavior of eigen values in differential operator with low parameter γ^{-1} at the senior derivative.

It is easy to discover the analogy of this problem with the so called quasiclassical approximation in quantum mechanics, where we are speaking of investigating Shredinger('s equation.

$$- h \psi'' + U(x) \psi = E \psi,$$

where U(x) is potential energy, E, total energy number, which has to determined, when Planck's constant h is taken as a low parameter. In our problem the potential energy is μ^2. The difference between Shredinger's equation and the equation (3.5) is that instead of the simple operator $\frac{d^2}{dx^2}$ we have operator L, which is included in Legendere's equation. This results in a certain additional complication, occurring in the presence of special points at the ends of segment (-1,1). Further on it will be seen, that the indicated difficulty could be overcome by means of a simple method. Using similarity of this problem to the problem of quasiclassical approximation, it is possible to apply the method called VKB-method, developed in that case. We shall mainly follow this method in the form suggested by Tsvaan{see Heding's book (1965) or the M.A. Evgrafov and M.V. Fedoryuk article (1966)}.

We shall analyse only the non-zonal case, $s \neq 0$, considering that the zonal one was investigated by L.N. Sretenskii (1947).

First of all we make conversion of equation (3.4). We denote $= \frac{I}{s}(1 - \mu^2)\frac{d\xi}{d\mu}$. Then instead of the one equation it is possible to write a system:

$$\xi' = \frac{s}{1 - \mu^2}\,\xi_1,$$

$$\xi'_1 = \frac{s}{1 - \mu^2}\,\xi + \frac{\gamma}{s}\,p\xi \quad , (p = \mu^2 - E).$$

We make another substitution of the sought for functions

$$\xi = \sqrt{\frac{s}{1 - \mu^2}}\,(z_1 + z_2), \quad \xi_1 = \sqrt{\frac{\gamma p}{s}}\,(z_1 - z_2) + \varphi(z_1 + z_2),$$

where

$$\varphi = \sqrt{\frac{1 - \mu^2}{s}}\left[-\frac{p'}{4p} + \frac{\mu}{2(1 - \mu^2)}\right]$$

It is easy to check, that for values, z_1, z_2 we will have a system

$$z_1' = \sqrt{\frac{\gamma p}{1 - \mu^2}}\,z_1 - \left[\frac{\mu}{2(1 - \mu^2)} + \frac{p'}{4p}\right] z_1 + \frac{s}{\gamma}\,\delta\,.(z_1 + z_2),$$

$$z_2' = -\frac{\gamma p}{1 - \mu^2}\,z_2 - \left[\frac{\mu}{2(1 - \mu^2)} + \frac{p'}{4p}\right] z_2 - \sqrt{\frac{s}{\gamma}}\,\delta\,.(z_1 + z_2).$$

(3.6)

Here $\delta(\mu)$ - a certain known function, which shall not wirte down, but will just mention, that it is regular everywhere, except values $\mu \doteq \pm 1$ and $\mu = \pm\sqrt{E}$ at which p converts into zero. In this case at the last points the feature has the order of $p^{-5/2}$.

Let us explain the meaning of the implemented replacement of variables. With accuracy up to residual term of about $O(1/\sqrt{\gamma})$ the system (3.6) divides into two individual equations

$$z_1' = \sqrt{\frac{\gamma p}{1-\mu^2}}\; z_1 - \left[\frac{\mu}{2(1-\mu^2)} + \frac{p'}{4p}\right] z_1,$$

$$z_2' = -\frac{\gamma p}{1-\mu^2}\; z_2 - \left[\frac{\mu}{2\,(1-\mu^2)} + \frac{p'}{4p}\right] z_2. \qquad (3.7)$$

Each of these integrates, without difficulty, and the solutions obtained should approximate the solutions of the complete system (3.6).

2. THE APPROXIMATING RESOLUTIONS.

If we denote

$$X_1(\mu) = \sqrt[4]{\frac{1-\mu^2}{p}}\; e^{\sqrt{\gamma}\;\eta(\mu)},\; X_2(\mu) = \sqrt[4]{\frac{1-\mu^2}{p}}\; e^{-\sqrt{\gamma}\;\eta(\mu)},$$

$$\eta(\mu) = \int_{\mu_0}^{\mu} \sqrt{\frac{p}{1-\mu^2}}\, d\mu,$$

(μ_0 - arbitrary point), then (3.7) has a fundamental system of solutions:

$$z_1^{(1)} = X_1,\; z_2^{(1)} = 0;\quad z_1^{(2)} = 0,\; z_2^{(2)} = X_2.$$

The attention is drawn to the circumstance that the approximating system (3.7), in distinction to complete system (3.6), has, besides the points $\mu = \pm\ 1$, also special points $\mu = \pm\sqrt{E}$. Correspondingly, for resolution (3.8) also these points are special (branching points). Thus, the regular function is approximated by ambiguous (Stokes') law. Obviously such approximation cannot take place uniformly throughout the whole range of variable μ. In order to find eigen values we must integrate the equation from -1 to 1. In the process of integration we must encounter points, in the vicinity of which the approximating resolution is useless. Here there are two ways for overcoming this difficulty.

The first occurs in special vicinity investigation of points $\mu = \pm\sqrt{E}$ with the object of finding approximation, applicable in these surroundings. The second way consists of the following: If the aim is not to obtain asymptotes of fundamental functions, but only of eigen values, then during the integration it is possible to come out on to a complex plane along the variable μ and to by-pass the special points $\mu = \pm\sqrt{E}$. This is the Tsvaan's method, which we shall use.

Let us investigate the behavior of functions (3.8) in the complex plane. The selection of point μ_0 is not essential, since with the variation of this point the resolution is multiplied by a constant. It is convenient to take as μ_0, one of the two points $\mu = \pm\sqrt{E}$. These points, from the analogy with quantum mechanics, we shall call the turning points. We exclude from analysis certain fixed surroundings of the ends of segment and origin of coordinates and, therefore, also the turning points, which strive toward zero at $\gamma \to \infty$, $f \to 0$. In Fig. 3.1 the thick lines show curves in plane , at which the function $\eta + = \int_{\sqrt{E}}^{\mu} \sqrt{\frac{\mu^2 - E}{1 - \mu^2}}\, d$ is purely imaginary.

Let us take function $\eta +$ in the region I+II. So that the function in this region would be unambiguous, we draw a section around the branching point $\mu = 1$ as shown in the figure. For the separation of the branch we shall agree to select the radical sign $\sqrt{(\mu^2 - E)(1 - \mu^2)}$ on the segment $[\sqrt{E}, 1]$ as positive. Then, as it is easy to see, function $\eta + (\mu)$ implements conformable reflection of region I+II on the region, shown in figure 3.2a. In this figure letters, confined within brackets denote points and lines, which are the images of points and lines denoted by the same letters in figure 3.1. Apparently, outline acd was reflected into twice passed imaginary negative semi-axis (line b reflects into positive semi-axis).

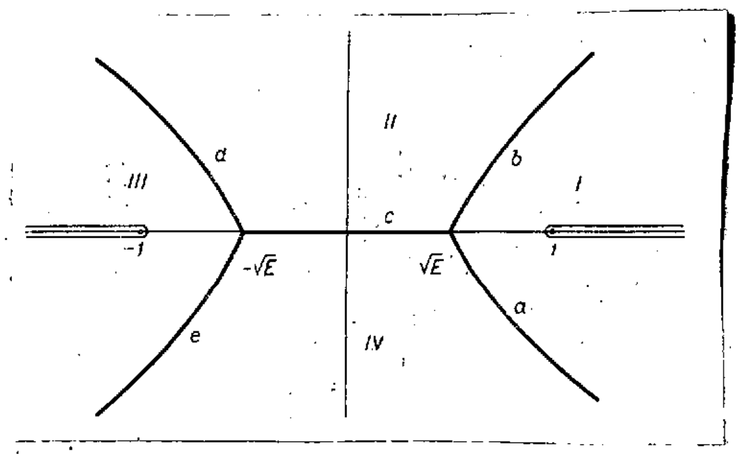

Fig. 3.1 – Plane of complex variable μ and of Stocks' line.

In the same way we analyse function $\eta^- = \int_{-\sqrt{E}}^{\mu} \sqrt{\frac{\mu^2 - E}{1 - \mu^2}}\, d\mu$.

The radical sign $\sqrt{(\mu^2 - E)/(1 - \mu^2)}$ on segment $[-1, -\sqrt{E}]$ we take as negative. This function reflects region II+III with section around $\mu = -1$ on to region, shown in figure 3.2b.

For us the important point now is one property of regions in figure 3.2. Any two poiints of such a region could be jointed by a curve, along which the real part of Reη represents a monotonous function. This property is quite obvious. In figure 3.2a one of such curves is shown by a dotted line. Passing on to prototypes, it may be said, that any two points of region I+II or II+III could be joined by a curve, along which Re η is monotonous.

We demonstrate an important lemma: if along a certain curve Re η is monotonous and the curve is at finite distance from special points, there are solutions of a complete system of equation (3.6), which on this curve are shown as

$$z_1 = X_1 \left[1 + 0 \left(\frac{1}{\sqrt{\gamma}} \right) \right], \quad z_2 = X_1 0 \left(\frac{1}{\sqrt{\gamma}} \right), \tag{3.9}$$

and also resolutions shown as

$$z_1 = X_2 0 \left(\frac{1}{\sqrt{\gamma}} \right), \quad z_2 = X_2 \left[1 + 0 \left(\frac{1}{\sqrt{\gamma}} \right) \right]. \tag{3.10}$$

assuming, that μ_1 is one of the ends of a curve, which is mentioned in the lemma's conditions. Taking system (3.6) as non-uniform and taking the residual terms, containing δ, as the known right portions, it will be possible to solve by the variation method of random constants, since we know solutions of the corresponding uniform equations. In this case we will have

$$z_1(\mu) = X(\mu) + \sqrt{\frac{s}{\gamma}}\; X_1(\mu) \int\limits_{\mu_1}^{\mu} \delta(\tilde{\mu}) \left[z_1(\tilde{\mu}) + z_2(\tilde{\mu}) \right] X_1^{-1}(\tilde{\mu})\, d\tilde{\mu},$$

$$z_2(\mu) = -\sqrt{\frac{s}{\gamma}}\; X_2(\mu) \int\limits_{\mu_1}^{\mu} \delta(\tilde{\mu}) \left[z_1(\tilde{\mu}) + z_2(\tilde{\mu}) \right] X_2^{-1}(\tilde{\mu})\, d\tilde{\mu}, \tag{3.11}$$

Here the random constants are chosen so that at $\mu = \mu_1$ solution coincides with that of the uniform system $z_1 = X_1$, $z_2 = 0$. The integration here is carried out along a curve having properties shown in condition of lemma. As μ_1 we take the end of the curve at which Re η has the lowest value. The (3.11) could be taken as a system of integral equations, which has replaced the system of differential equations.

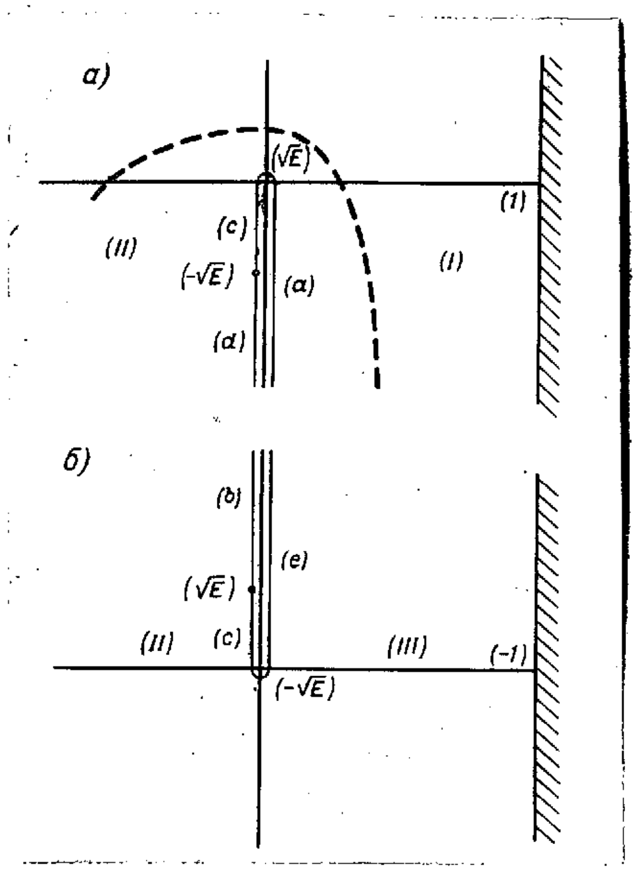

Fig. 3.2 - Conformable reflection of plane, given by functions $\eta^+(\mu)$[a] and $\eta^-(\mu)$[b].

We shall resolve this system by expanding functions into series according to orders of $1/\sqrt{\gamma}$, i.e. assuming that

$$z_1 = \sum_{n=0}^{\infty} \left(\gamma\sqrt{\frac{s}{\gamma}}\right)^n z_1^{(n)}, \quad z_2 = \sum_{n=0}^{\infty} \left(\gamma\sqrt{\frac{s}{\gamma}}\right)^n z_2^{(n)}. \tag{3.12}$$

For the factors of a series we get recurrent relations

$$z_1^{(0)} = X_1,$$

$$z_1^{(n)} = X_1(\mu) \int_{\mu_1}^{\mu} \delta(\tilde{\mu}) \left[z_1^{(n-1)}(\tilde{\mu}) + z_2^{(n-1)}(\tilde{\mu}) \right] X_1^{-1}(\tilde{\mu})\, d\tilde{\mu},$$

$$z_2^{(0)} = 0$$

$$z_2^{(n)} = -X_2(\mu) \int_{\mu_1}^{\mu} \delta(\tilde{\mu}) \left[z_1^{(n-1)}(\tilde{\mu}) + z_2^{(n-1)}(\tilde{\mu}) \right] X_2^{-1}(\tilde{\mu})\, d\tilde{\mu}.$$

Now it is easy to obtain evaluation from induction

$$\left| z_1^{(n)} \right| < K^n \left| X_1 \right|,$$

$$\left| z_2^{(n)} \right| < K^n \left| X_1 \right|,$$

where as constant K it is possible to take product 2 max$|\delta|$ by length of integration curve. Ineed, at n = 0, this evaluation, obviously, holds true, If it holds true for n - 1, then

$$\left| z_1^{(n)} \right| \leqslant 2 \max |\delta| \, |X_1| \, K^{n-1} \int_{\mu_1}^{\mu} |d\tilde{\mu}| = K^n |X_1|,$$

$$\left| z_2^{(n)} \right| \leqslant 2 \max |\delta| \left| \sqrt[4]{\frac{1-\mu^2}{p}} \right| e^{-\sqrt{\gamma}\,\mathrm{Re}\,\eta(\mu)} \times$$

$$K^{n-1} \int_{\mu_1}^{\mu} e^{2\sqrt{\gamma}\,\mathrm{Re}\,\eta(\tilde{\mu})} |d\tilde{\mu}| \leqslant$$

$$\leq K \left| \sqrt[4]{\frac{1-\mu^2}{P}} \right| e^{\sqrt{\gamma}\ \mathrm{Re}\ \eta(\mu)} K^{n-1} = K^n \ |X_1|$$

In the last case we had to use the fact that Re $\eta(\tilde{\mu}) <$ Re$\eta(\mu)$, i.e., the property of monotony indicated in the lemma condition. From the evaluation obtained it follows, that the series (3.12) absolutely converge with sufficiently high γ and that there is actually (3.9). The second half of assertion, pertaining to formula (3.10) is proved similarly, but the integration here should be done from the end of the curve, where Re η has the highest value.

Inasmuch as each solution of the system is a linear combination of two linearly independent solutions, for instance, of those the existence of which is asserted by the proved lemma, any solution of system (3.6) could be written as

$$Z_1 = C_1X_1 \left[++ \ 0 \left(\frac{1}{\sqrt{\gamma}}\right) \right] + C_2X_2 0 \left(\frac{1}{\sqrt{\gamma}}\right),$$

$$Z_2 = C_1X_1 0 \left(\frac{1}{\sqrt{\gamma}}\right) + C_2X_2 \left[1 + 0 \left(\frac{1}{\sqrt{\gamma}}\right) \right].$$

Returning to initial variables, it is possible to come to the conclusion, that any solution of ξ could be written on our curve as

$$\xi = \left\{ C_1(\gamma)X_1 \left[1 + 0 \left(\frac{1}{\sqrt{\gamma}}\right) \right] + C_2(\gamma)\ X_2 \left[1 + 0 \left(\frac{1}{\sqrt{\gamma}}\right) \right] \right\} \frac{s}{1-\mu^2}$$

$$\xi \sim \left\{ C_1(\gamma)X_1(\gamma,\mu) + C_2(\gamma)\ X_2(\gamma,\mu) \right\} \sqrt{\frac{s}{1-\mu^2}} \qquad (3.13)$$

N.B. It is easy to see, that concepts (3.9) and (3.10) hold true not only on some fixed curve, but throughout the region, each two points of which could be joined by a curve having the above property of monotony, atleast, if we restrict ourselves to some terminal portion of this region and exclude the surroundings of special points.

3. CONJUGATION OF ASYMPTOTIC FORMULAS IN A COMPLEX PLANE :

Assuming now that ξ – is the proper solution of our problem, i.e., solution of equation (3.5), satisfying limiting conditions at $\mu = \pm 1$, the ends of segment are single valued equation, therefore, $\xi(\mu)$ is analytical function of a complex variable μ. In accordance with the facts proved, in region I + II, except the above indicated surroundings, ξ admits approximation.

$$\xi \sim \left[(1-\mu^2)\,(\mu^2-E)\right]^{-1/4}\left[C_1(\gamma)\,e^{\sqrt{\gamma}\,\eta+(\mu)} + C_2(\gamma)\,e^{-\sqrt{\gamma}\,\eta+(\mu)}\right]. \tag{3.14}$$

What can be said regarding factors C_1 and C_2? These should be determined by boundary conditions $\xi = 0$ at $\mu = 1$. But, unfortunately, the asymptotes do not pertain to surroundings of point $\mu = 1$. However, this difficulty could be overcome without the search for asymptoes, suitable for surroundings of point $\mu = 1$. First of all for the solution which we are speaking about, throughout the segment $[\sqrt{E}, 1]$ there should be $\xi'/\xi < 0$. This follows from equation (3.5) for function ξ, if it is re-written as

$$(1-\mu^2)\,\xi'' = 2\mu\,\xi' + \frac{s^2}{1-\mu^2}\,\xi + \gamma(\mu^2 - E).$$

If at a certain point of segment ξ' and ξ had similar signs, for instance positive, then, since $\xi(1) = 0$, a point would have been found on segment $[\sqrt{E}, 1]$, of maximum, at which $\xi' = 0$, $\xi = 0$. But in the equation just wiritten we obtain at this point $\xi'' > 0$, which cannot be at maximum point.

On segment $[\sqrt{E}, 1]$ the first of exponent in formula (3.14) increases with the increase of γ, whereas the second decreases. In order to fulfill the condition $\xi'/\xi > 0$ at any γ, the increasing exponent should be suppressed at the cost of the factor, i.e., the factor with increasing component should be infinitely low in comparison with the factor with attenuating exponent, i.e., $C_1(\gamma)/C_2(\gamma) \rightarrow 0$, in which case the striving to zero is exponential. At segment $[\sqrt{E}, 1]$ the main term should be real. Therefore, C_2 is a real number with accuracy atleast to the term, which strives toward zero in comparison with C_2.

In exactly the same way it is possible to write approximation formula in the region III + II (see figure 3.1).

$$\xi \sim \frac{}{\sqrt[4]{(1-\mu^2)(\mu^2 - E)}} \left[d_1(\gamma)\, e^{\sqrt{\gamma}\,\eta^-(\mu)} + d_2(\gamma)\, e^{-\sqrt{\gamma}\,\eta^-(\mu)} \right] \tag{3.15}$$

Once again the exponent increasing on segment $[-1, -\sqrt{E}]$ should be suppressed, i.e., $d_1(\gamma)/d_2(\gamma) \rightarrow 0$. And here also d_2 should be real.

Asymptotic formulas (3.14) and (3.15) have common region of application sector II. Here there should be coincidence of atleast the main terms of these asymptotic expansions. Which are the main terms? Those which were getting attenuated in sectors I and III, increase in sector II with the γ increase. Moreover their factors C_2 and d_2, as we saw, are greater than the factors of other terms. Therefore, it is precisely these terms that have to be equated one to the other:

$$\frac{1}{\sqrt[4]{(1-\mu^2)\,(\mu^2-E)}}\,C_2 e^{-\sqrt{\gamma}\int\limits_{\sqrt{E}}^{\mu}\sqrt{\frac{\mu^2-E}{1-\mu^2}}\,d\mu} =$$

$$= \frac{1}{\sqrt[4]{(1-\mu^2)\,(\mu^2-E)}}\,d_2 e^{\sqrt{\gamma}\int\limits_{\sqrt{E}}^{\mu}\sqrt{\frac{\mu^2-E}{1-\mu^2}}\,d\mu}$$

Moreover, it should be taken into account that multipliers $1/\sqrt[4]{(1-\mu^2)\,(\mu^2-E)}$, which stand to the left and right, are equal to each other in absolute value, but are distinct in phase, i.e., different branches of radicals have to be taken. In the left portion of the equation we take the branch which is real and positive at $\sqrt{E} < \mu < 1$, and in the right that which is real and positive at $-1 < \mu < -\sqrt{E}$. We obtain

$$e^{-i\,\pi/2}\,\frac{C_2}{d_2} = e^{\sqrt{\gamma}\int\limits_{-\sqrt{E}}^{\sqrt{E}}\sqrt{\frac{\mu^2-E}{1-\mu^2}}\,d\mu}$$

The radical under the sign of integral in the exponent is purely imaginary. We have

$$\frac{C_2}{d_2} = e^{i\left(\frac{\pi}{2} - \int_{-\sqrt{E}}^{\sqrt{E}} \sqrt{\frac{E - \mu^2}{1 - \mu^2}}\, d\mu\right)}$$

Taking from here the imaginary part and considering the reality of factors C_2 and d_2, we get

$$\sin\left(\frac{\pi}{2} - \sqrt{\gamma} \int_{-\sqrt{E}}^{\sqrt{E}} \sqrt{\frac{E - \mu^2}{1 - \mu^2}}\, d\mu\right) = 0,$$

$$\sqrt{\gamma} \int_{-\sqrt{E}}^{\sqrt{E}} \sqrt{\frac{E - \mu^2}{1 - \mu^2}}\, d\mu = \pi\left(p + \tfrac{1}{2} \right). \qquad (3.16)$$

where p is a certain whole number. This is the sought for relation between γ and E. It is completely analogical to Bohr's quantization rule for Shredinger's equation of quantum mechanics in quasiclassical approximation, for low values of Planck's constant h.

From the given demonstration it is easily discernible, that argument variation of function ξ, with by-pass along the outline in the top semi-plane from some point on segment $[\sqrt{E}\ , 1]$ to some point on segment $[-1, -\sqrt{E}]$ is equal to $\sqrt{\gamma} \int_{-\sqrt{E}}^{\sqrt{E}} \sqrt{(E - \mu^2)/(1-\mu^2)}\, d\mu - \frac{\pi}{2}$ i.e., according to formula (3.16), to 2 p. Therefore, with complete by-pass in the upper and lower half-plane the argument variation is equivalent to $2\pi p$. In other words number p is interpreted as a number of zeros of function inside the outline, which by-pass segment $[-\sqrt{E}, \sqrt{E}]$. Most probaly all these zeros lie on this segment. But even without knowing it, we may hence draw some conclusions from the proven fact regarding conjugation of asymptotes at high and low γ^{-1}. It is obvious, with even p function ξ is even, since it has an even number of zeros, and vice-versa.

4. <u>TWO BRANCHES OF ASYMPTOTES</u> :

Asymptotes in form (3.16) are not very convenient for use, as they contain an integral, not calculated in elementary function (elliptical). However, the formula could be appreciably simplified, if the fact is noted, that out of $\gamma \to \infty$, according to this formula, it follows, that $E \to 0$ (for one and the same p, i.e,, for one and the same mode). In this case the integration interval gets contracted and in denominator of integrand it is possible to discard ^{2} as opposed to unit, which will not affect the main term of the asymptotes. In this way there is practically always a very high accuracy; it could have been increased, if desired, by expanding the integral according orders of E and taking two expansion terms. But now the integral is calculated, and we have

$$\sqrt{\gamma}\; E = 2p + 1 \; . \tag{3.17}$$

Now we recall, what is E. We find

$$\sqrt{\gamma}\left(f^2 + \frac{s}{f_\gamma}\right) = 2p + 1. \tag{3.18}$$

This equation could be solved as quadrant in relation to $\sqrt{\gamma}$. We will get two radicals, i.e., two branches of solution. Since $f \to 0$, the terms obtained could be expanded according to orders of f. Thus, we arrive at the formula

$$f^2\gamma^{1/2} \sim 2p + 1 - \frac{sf}{2p + 1} \qquad (3.19)$$

for the first branch of the solution and formula

$$f\gamma^{1/2} \sim \frac{s}{2p + 1} \qquad (3.20)$$

for the second branch. In formula (3.19) we have taken two terms of expansion according to orders of f, and in formula (3.20) only one. As shown by the calculation experience, this is found to be quite sufficient.

We see that according to behavior at low γ^{-1} the characteristic curves also get divided into two groups with curves of first order on the left (at low γ^{-1}) and the curves of the second order on the right. The question remains open, as to whether division into curves of first and second order on the left at low γ^{-1} corresponds to similar division on the right at high γ^{-1}. In other words, whether the curve of the first order on the left is the same one on the right and vice versa. The answer to this question cannot be obtained from a single asymptote. It depends to a considerable extent on the behavior of solutions in the intermediate zone, between the asymptotes. Let us take a look on the results of numerical calculations, shown in figure 2.5. We know that the continuous lines here show characteristic curves, calculated on electronic computer, and the short dotted lines asymptotes on the right at high γ^{-1}. Long dashes show curves, depicted by equations (3.19) and (3.20). The figures show, that the asymptotic formulas give a very good approximation. At considerable extention the asymptotic curves practically merge with the exact curves. In this case the activity zone of asymptotes at high γ^{-1} almost merges with their activity zone at low γ^{-1}. We may say, that asymptotic formulas alone are sufficient to reconstruct satisfactorily the shape of curve throughout its extent. Further on we discover, that there is no reciprocal unambiguous correspondence between the classes of curves on the right and left. In all the calculated examples at s = 1, 2, 3, there is the same occurrence: the bottom curve of the second order on the right (at n = s in formula (2.21) is at the same time the top one of the first order curves on the left (at p = 0 in formula (3.19)) . The remaining curves pertain to similar classes both on the left and right. In para 7 it will be shown, that this is in general conformity to principle.

It would be of interest also to write down formulas for periods in all the asymptotic cases in dimensional form. For asymptotes at high γ^{-1} for the curves of the first and second order respectively we will have

$$T \sim \frac{2\pi a}{\sqrt{gh}} \sqrt{n(n+1)}, \quad T \sim \frac{n(n+1)}{2s} \tag{3.21}$$

For asymptotes at low γ^{-1} we have

$$T \sim \frac{\pi}{\sqrt{2p+1}} \sqrt{\frac{2a}{\omega\sqrt{gh}}}, \quad T \sim \frac{2\pi a}{\sqrt{gh}} \frac{2p+1}{s} \tag{3.22}$$

for curves of the first and correspondingly second order. These formulas realize all three possibilities of the plotting of parameter, having dimension of time, from the values a, ω and $\sqrt{gh}$

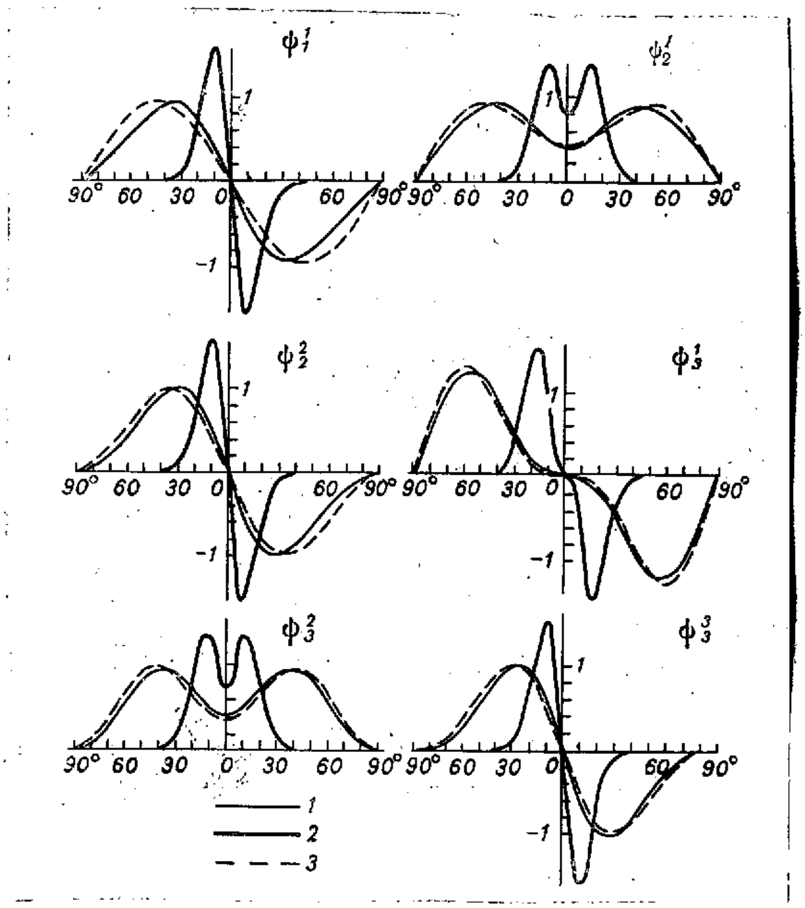

Fig.3.3. - Fundamental functions of the Laplace's tidal equation 1) γ^{-1}=1/10, 2) γ^{-1}=1/1200, 3) asymptotics $\gamma^{-1} \to \infty$.

Simultaneously with eigen f values we calculate also the fundamental functions of ψ tidal equation, i.e. Hough's function. At high γ^{-1}, as we know, these functions are asymptotically equal to adjoined Legendre functions (for asymptotes of the first order) or linear combination of two such functions (for asymptotes of the second order). With decrease of γ^{-1} Hough's functions become dombinations of a high number of adjoined Legendre functions and their form becomes more complex. Plotted in figure 3.3. are the curves of calculated Hough's functions of the second order for some s and n. We see, that at γ^{-1}=1/10 the asymptotes are still accurate, whereas at γ^{-1}=1/1200 the curves acquire a unique nature - the whole oscillation amplitude is concentrated near the equator, close to $\mu = 0$, whereas in the zones of moderate and high latitudes practically turns into zero. Such behavior of natural oscillations is easily explained. Outside the turning points the solution exponentially dies-out. The turning points are determined by formula $\mu = \pm\sqrt{E}$, or $\mu = \pm\sqrt{f^2 + s/f\gamma}$, which for asymptotes of the second order could be written also as $\mu \approx \pm\sqrt{s/f\gamma}$. For instance, for s = 1, γ = 1200, f = 1/100 we shall have = $\pm$ 0.29. Outside the interval 0.29 $< \mu <$ 0.29 the resolution exponentially dies out.

In para 7 of chapter 2 it was indicated, that at f 1 there is appearance, at the ends of segment $[-1, 1]$, of zones where the solution of ψ (exact) cannot oscillate. Using the asymptotes obtained we can implement a more exact evaluation of the width of these zones. For instance, it is easy to check that for asymptotes both of the first and second order (in the last case only at p $\geqslant$0) the factor in front of ψ in the right portion of (2.25) is positive at (1 + ε)$\sqrt{E} < \mu < 1$, where ε is random positive number. Therefore, in this interval there are no zeros of function ψ.

The fundamental functions at high γ^{-1} are used in the theory of diurnal tides, (see Haurwitz, 1965).

5. CORRECTNESS OF CARRIED OUT APPROXIMATION. ADDITIONAL WAVE, PROPAGATING EASTWARD:

At the start of our discussion we made one, although very plausible, but not very well-founded, assumption according to which the role of term 2 $\mu\gamma\psi$ in equation (3.1) is insignificant so that it could be disregarded. As a result we obtained equation (3.5), to which we applied the VKB-approximation technique. Investigation should have been made of the validity of this first simplification, by comparing the exact solutions of equation (3.5) and the initial Laplace system (2.3). Unfortunately, it cannot be done very exactly. It is only possible to state certain reasons,which could be considered as eurystic in regard to validity of the first approximation. We turn our attention to this question now, because the answer is not quite insignificant. There is one exceptional case; when the resolution of equation (3.5) does not correspond to any solution of exact system and, vice-versa, the exact system has a soluton undepictable by equation (3.5). In the present paragraph we shall name as approximating solution that of equation (3.5), and exact solution of system (2.3).

It should be mentioned that even if it is assumed that the approximating solution ζ is uniformly similar, jointly with derivative to exact resolution, it, nevertheless, will not have one fine structural property of this solution. Namely, at the critical points of the second order, i.e., at those points on axis where the term $(S^2/f^2) - (1 - \mu^2)\gamma$ converts into zero, conversion into zero should happen also to linear combination $(1 - \mu^2)\zeta' + s\mu\zeta/f$, as follows from the second equation of system (2.3). But the asymptotic solution does not meet this condition. We mention, that for asymptotes of second order at $\rho > 0$, i.e., $f^2\gamma \sim s^2/(2p+1)^2$, critical points of the analysed type do not exist at all. Whereas for asymptotes of the first order, i.e, $f^4\gamma \sim (2p+1)^2$, these points are present near the ends of segment $[-1, 1]$ and at $\gamma \to \infty$ they strive toward the ends. This fineness becomes important if from function , determined from equation (3.5), we wish to find ψ by means of the second equation (2.3). At the critical point ψ will be found to be infinite.

To evaluate, to some extent, the effect of discarded term $2\mu\gamma\psi$ would be possible with the use of perturbation theory and considering this term as perturbation. If the first correction is found to be asymptotically low, we have some grounds to assume our approximation as justified. Keeping in view the above indicated difficulty, bound with the existence of critical points near the ends of segment $[-1, 1]$, we shall slightly narrow down this segment, taking instead $[-1+\varepsilon, 1-\varepsilon]$, where the ε will be fixed. At the ends of the segment we set zero conditions. It may be assumed that this substitution should not be felt appreciably, at least the main term should be obtained correct, since our solutions quickly die-out at the ends of the segment, and the higher γ is, the quicker is the vanishing. We must demonstrate now, that if at $\gamma \to \infty$ there is a set of resolutions of approximating equation (3.5), for which $f^2\gamma \sim s^2/(2p+1)^2$, $p > 0$ or $f^4\gamma \sim (2p+1)^2$, $p \geqslant 0$, there is also the corresponding set of resolutions of the exact system; moreover, f (γ) being asymptotically similar to this set for exact resolution, and vice-versa, each resolutions set for exact system in these conditions corresponds to the set of near resolutions of approximating equation.

If in equation (3.1) no terms are disregarded, then with the same conversions, as in para 1, instead of (3.5) we shall have

$$L\zeta + \gamma (E - \mu^2)\zeta = 2\mu\gamma\psi.$$

The right portion we estimate as perturbance, assuming, that the unperturbed resolution, which we shall mark by index zero, meets the previous equation, where there was no additional term

$$L\zeta_o + \gamma (E_o - \mu^2)\zeta_o = 0.$$

Expanding according to the orders of low parameter, which could provisionally be set in front of perturbing term, we shall have for the first order terms

$$L\zeta_1 + \gamma (E_o - \mu^2)\zeta_1 + \gamma E_1\zeta_o = 2\mu\gamma\psi_o.$$

Multiplying the first equation by ζ_1, the second by ζ_o, subtracting and integrating, we shall

$$E_1 = \frac{2\int_{-1+}^{1-} \mu\psi_o\zeta_o d\mu}{\int_{-1+}^{1-} \zeta_o^2 d\mu}$$

Here $\psi_o = \left[(1 - \mu^2)\xi_o - s\mu\xi_o/f\right] / \left[(s^2/f^2) - (1-\mu^2)\gamma\right]$
The term $(s^2/f^2) - (1 - \mu^2)\gamma$ is evaluated from bottom as $c\,\gamma$ on our assumptions regarding the bond of f and γ. Now we must estimate individually two terms

$$\int_{-1+\varepsilon}^{1-\varepsilon} \frac{\mu(1-\mu^2)\xi_o\xi_o}{\frac{s^2}{f^2} - (1-\mu^2)\gamma} d\mu ,$$

$$s\int_{-1+\varepsilon}^{1-\varepsilon} \frac{\mu^2\xi_o^2}{f\left[\frac{s^2}{f^2} - (1-\mu^2)\gamma\right]} d\mu .$$

In the first of these $\xi_o\xi_o$ has to be substituted by $(\xi_o^2)^{1/2}$ and integrated by parts. Without difficulty we get estimate $O(\gamma^{-3/2})$. In the second we get initially estimate

$$\frac{c\displaystyle\int_{-1+\varepsilon}^{1-\varepsilon} \frac{\mu^2\xi_o^2}{f\gamma} d\mu}{\displaystyle\int_{-1+\varepsilon}^{1-\varepsilon} \xi_o^2\, d\mu} .$$

Function ξ_o dies-out as exp. $(-\sqrt{\gamma}\,\mu)$, [This follows from asymptotic formula (3.14)] . Therefore, this entire term is estimated as $O(\gamma^{-2})$. Thus, we have $E_1 = O(\gamma^{-3/2})$, whereas $E = O(\gamma^{-1/2})$. The correction is asymptotically low.

Thus, we have grounds for assumption, that the asymptotic solutions, which we have obtained actually correspond to the true resolutions in every case, except, perhaps, $p = 0$ for waves of the second order, where the above reasoning is not applicable.

Now let us pay attention to the fact that we have obtained our asymptotes, using equation (3.1), i.e., the first of system (2.8) equations. But the second equation could have been used just as well and the same carried out, i.e., the two discarded as against c s/f. The result would have been equation

$$\left(L - \frac{S}{f}\right)\psi + \gamma\ \left(f^2 - \mu^2\right)\psi\ = 0$$

instead of (3.5). The difference is that instead of function ζ we have ψ , and f is included with an opposite sign. Therefore, if in relation to this equation the same asymptotic theory is developed, we will obtain the same formulas for asymptotes of the first order (at least in the main term, not dependent on sign f), but only now at even p the even functions will be not ρ , but ζ .ψ For asymptotes of the second order we will get instead of (3.20) formulas with opposite sign $f\gamma^{1/2} \sim -s \sim s(2p+1)$. But all these asymptotic solutions cannot be near the exact solutions, i.e. cannot have meaning, except, perhaps, asymptotic solution of second order at $p = 0$, i.e.,

$$f\gamma^{1/2} \sim - S,$$

because in other cases, as we have defined, the exact resolution is similar to solutions of equation (3.5), i.e., our old asymptotic formulas, which contradict new formulas, still hold true. In para 7 we will see, that for branch $f\,\gamma^{1/2} \sim \pm S$ the correct sign is actually the minus.

We make one more comment regarding the number of zeros of the exact and approximating solutions ξ. All zeros of approximating solution ξ lie, as we know, on the narrow segment $[-\sqrt{E},\ \sqrt{E}]$, which contracts toward zero at $\gamma \longrightarrow \infty$. If the approximating solution is uniformly close to the exact, then in the vicinity of its zeros should lie zeros of the exact solution and in the same quantity. But besides this, in exact solution zeros may appear also within the surroundings of the ends of the segment $[-1, 1]$. It is quite clear that in the case of $f < 0$ the exact solution has two additional zeros. Whereas, in the case of $f > 0$ these zeros cannot be present and, therefore, the number of zeros in exact and approximating solution is the same. In fact, at the end of preceding paragraph we remarked, that outside the interval $[-2\sqrt{E},\ 2\sqrt{E}]$ function Ψ (the exact) does not convert into zero. Assuming, that in interval $[2\sqrt{E},\ 1]$ the sign of Ψ will be positive, at $\mu = 2\sqrt{E}$ the right portion of the equation (2.3)

$$(1-\mu^2)\xi' + \frac{s\mu\xi}{f} = \left[\left(\frac{s^2}{f^2}\right) - (1-\mu^2)\right]\Psi$$

is negative, and at $\mu = 1$ positive. If it is taken that the exact solution approximates smoothly at $\mu = 2\sqrt{E}$, then at this point function ξ, and its derivative have different signs. These signs could only be: $\zeta < 0, \zeta > 0$. But in the immediate vicinity of $\mu = 1$ function ζ, monotonuously strive toward zero, ζ and ζ', also have different signs: $\zeta > 0, \zeta < 0$. Therefore, function ζ changes sign at segment $\left[2\sqrt{E}, 1\right]$. Demonstration of the fact that there cannot be more than one zero, is easily done by means of similar reasoning with additional consideration, that between two zeros of function ζ there has to be either ψ zero (which is not present here), or a critical point of the second order. The proof is simple, as we omit it. The proff of the fact, that at $f > 0$ there are no ζ zeros on segment $\left[2\sqrt{E}, 1\right]$ is that at $0 < f < 1$ the extreme, nearest to ends, are the ψ zeros, and not ζ, which ensues from the reasoning of para 6 in Chapter 2.

6. NEGATIVE VALUES OF EQUIVALENT DEPTH:

We shall specially pause on one more case, which has lately attracted the attention of some investigators: the case of negative values of equivalent depth h or parameter γ. We know, that these values could only be at $f < 1$ and that with increase of f^{-1} the curves, one by one, separate and move away to horizontal asymptotes (in the case of positive f), to which already in positive half-plane $\gamma < 0$ arrive the curves of second order (see Fig: 2-1, 2-2). But so far we do not know, whence these curves ensue, from which points on the axis of ordinates. Perhaps the whole bunch of these curves emerges from one point $f = 1$, which at a glance at the drawing seems quite probable, and, perhaps, each curve emerges from its own point on the axis of ordinates. In other words, the behavior of these curves should be investigated at low negative γ^{-1} values.

Further on, it will be proved that the characteristic curves virtually emerge from one point f = 1 at $\gamma^{-1} = 0$. This only requires to prove, that an infinite number of curves emerges from this point. If in these conditions at least one curve had emerged from another point on the axis of ordinates, it would have been interesecting with some of the curves, emerging from point f = 1, since the curves can accumulate only toward the axis of ordinates, and there cannot be intersection. Hence we shall analyse only f values close to unity and plot multiple of curves, striving toward point f = 1 at $\gamma^{-1} \longrightarrow 0$.

Asymptotes, obtained in the preceding paragraph, and the methods of obtaining them are inapplicable in the present case, since there we used not only the lowness of γ^{-1}, but also the lowness of f, which is not present here. It will be necessary to obtain a new working system of equations for the use of VKB-method.

$$\psi = \sqrt{\frac{fp^3}{S(1 - \mu^2)}}\,(z_1 + z_2)$$

$$\zeta = \sqrt{\frac{-\gamma\, pf}{s}}\,(z_1 - z_2) + \varphi\cdot(z_1 + z_2), \qquad (3.23)$$

where

$$p = \sqrt{f^2 - \mu^2}, \quad \varphi = \frac{1}{2}\sqrt{\frac{f(1-\mu^2)}{sp}}\left[\frac{p'}{p} + \left(1 - \frac{2s}{f}\right)\frac{\mu}{1 - \mu^2}\right].$$

It is easy to check that instead of system (2.3) we shall have in these variables the following system:

$$z'_{1,2} = \pm \sqrt{-\gamma}\,\frac{p}{\sqrt{1-\mu^2}}\,z_{1,2} - \left(\frac{\mu}{2(1-\mu^2)} + \frac{p'}{p}\right)z_{1,2} \pm$$

$$\pm \frac{\delta}{\sqrt{-\gamma}}\,(z_1 + z_2) \qquad (3.24)$$

where δ is a certain function universally regular, except at points $\mu = \pm 1$ and $\mu = \pm f$. The top signs pertain to equation for z'_1 and the bottom ones to equation for z'_2. The role of this system is the same as of system (3.6), it separates (with accuracy up to residual term of about $\frac{1}{\sqrt{-\gamma}}$) into two individual equations, each of which integrates without difficulty. Thus, system (3.24) is approximated by system

$$z'_{1,2} = \pm \sqrt{-\gamma}\,\frac{p}{\sqrt{1-\mu^2}}\,z_{1,2} - \left(\frac{\mu}{2(1-\mu^2)} + \frac{p'}{p}\right)z_{1,2},$$

resolutions of which

$$z_1 = \frac{\sqrt[4]{1-\mu^2}}{\sqrt{f^2-\mu^2}}\,e^{\sqrt{-\gamma}\int^{\mu}\sqrt{\frac{f^2-\mu^2}{1-\mu^2}}\,d\mu}, \qquad z_2 = 0;$$

$$z_1 = 0,\ z_2 = \frac{\sqrt[4]{1-\mu^2}}{\sqrt{f^2-\mu^2}}\,e^{-\sqrt{-\gamma}\int^{\mu}\sqrt{\frac{f^2-\mu^2}{1-\mu^2}}\,d\mu}$$

We shall name the VKB-solutions. In old variables the VKB-solutions have the following appearance:

$$\Psi = \sqrt[4]{\frac{f^2-\mu^2}{1-\mu^2}}\,e^{\pm\sqrt{-\gamma}\int^{\mu}\sqrt{\frac{f^2-\mu^2}{1-\mu^2}}\,d\mu}$$

(the residual terms are not written).

It has to be investigated, in which zone of complex plane the VKB-solutions approximate solutions of a complete system. The lemma, similar to the one proved above holds true even here: any resolution could be approximated by linear combination of VKB-solutions

$$\psi = \sqrt[4]{\frac{f^2 - \mu^2}{1 - \mu^2}} \left(C_1(\gamma)\, e^{\sqrt{-\gamma} \int^{\mu} \sqrt{\frac{f^2 - \mu^2}{1 - \mu^2}}\, d\mu} + C_2(\gamma)\, e^{-\sqrt{-\gamma} \int^{\mu} \sqrt{\frac{f^2 - \mu^2}{1 - \mu^2}}\, d\mu} \right) \tag{3.25}$$

in the zone of complex variable μ, where any two points could be joined by an outline, along which Re η ($\eta = \int_f^{\mu} \sqrt{-\frac{f^2 - z}{1 - \mu^2}}\, d\mu$) is monotonous.

Since we are analysing a case of f, near to unity, it is possible to encircle both the points f and the unity by a common fixed surrounding, into which we shall not enter. Hence we shall examine separately cases of even and odd solutions, which meet at $\mu = 0$ boundary conditions

$$\psi' = 0 \quad \text{and} \quad \psi = 0 \tag{3-26}$$

respectively. Therefore, the analysis will be carried out only of special points $\mu = f$, $\mu = 1$. In Fig: 3-4 thick lines show curves, at which Re $\eta = 0$, i.e., the Stockes' lines. If a cut is made along an outline, shown in Fig: 3-4 by an arrow, the plane μ will be reflected throughout plane (Fig: 3.5), cut along a negative imaginary semiaxis (the sign of radical was selected so that at real $\mu < f$ the radical is positive, and η, therefore, is negative). From what has been said above it is clear that point $\mu = 0$ could be joined with point μ_o, where μ_o is random fixed point, lying to the right of point $\mu = 1$, by an outline, along which the Re η is monotonuous. This outline is shown in the figure by a dotted line. On this outline the ψ solution is approximated by VKB-solution.

The question arises, how to estimate the boundary condition at $\mu = 1$. In the vicinity of this point the VKB-solutions are ineffective. Here we are assisted by the reasoning, that boundary condition conversions of into zero at $\mu = 1$ signifies also, that the solution at this point is a product $(1 - \mu^2)^{s/2}$ by analytical function, in distinction from other solutions, which have at this point logarithmic branching. Thus, if analysis is made of the real part of the solution [10, 11] then with even value of s it will be real even at $\mu > 1$, and with odd value of s purely imaginary at $\mu > 1$. For the other solutions, not meeting boundary conditions, this is not so.

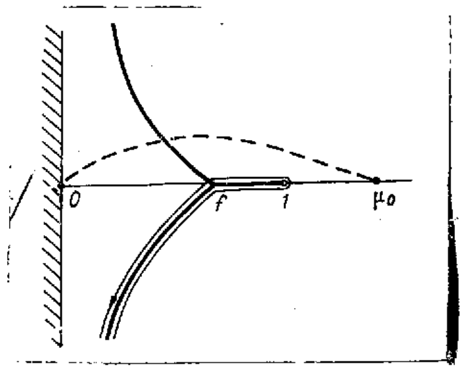

Fig: 3-4 - Plane of composite variable μ.

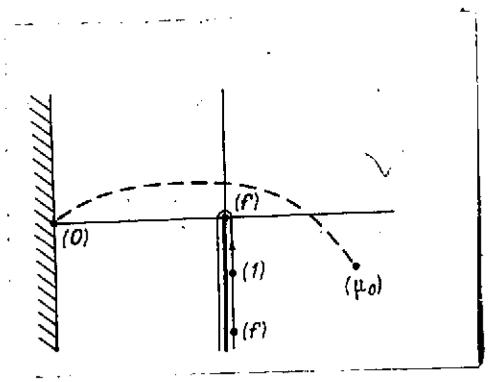

Fig: 3-5 - Conformable reflection of μ, given by function $\eta\ (\mu)$.

Thus, the boundary condition at $\mu = 1$ is substituted by the following: with even s the solution should be real at $\mu > 1$, and with odd s purely imaginary.

We denote for simplicity

$$\rho = \sqrt{-\gamma}\ \sqrt{\frac{f^2 - \mu^2}{1 - \mu^2}}\ .$$

Let us analyse individually the cases of even and odd solutions of ψ. Case 1 even solutions of ψ. Due to boundary condition at zero (3-26) our VKB-approximation (3.25) should have the following appearance

$$\psi \sim \sqrt{\rho}\,\mathrm{ch} \int_0^{\mu} \rho\, d\mu,$$

or

$$\psi \sim \sqrt{\rho}\,\mathrm{ch}\left[\int_0^{f} \rho\, d\mu + \int_f^{\mu} \rho\, d\mu\right].$$

At $\mu = \mu_o$ it gives

$$\psi(\mu_o) \sim \sqrt{\rho} \quad \mathrm{ch}\left[\int_0^{f} \rho\, d\mu + \int_f^{1} \rho\, d\mu + \int_1^{\mu_o} \rho\, d\mu.\right]$$

Integrals $\int_0^{f}$ and $\int_1^{\mu_o}$ are real and integral $\int_f^{1}$ purely purely imaginary. The formula just written could be changed, separating the real and imaginary parts,

$$\psi(\mu_0) = \sqrt{p}\left\{ \mathrm{ch}\left[\int_0^f p\,d\mu + \int_1^{\mu_0} p\,d\mu\right] \cos\int_f^1 |p|\,d\mu - \right.$$

$$\left. i\,\mathrm{sh}\left[\int_0^f p\,d\mu + \int_1^{\mu_0} p\,d\mu\right] \sin\int_f^1 |p|\,d\mu \right\}.$$

Now we shall use the second boundary condition.

1. Assuming s is even, then $\psi(\mu_0)$ should be real.

This gives

$$\sin\int_f^1 |p|\,d\mu = 0.$$

or

$$\sqrt{-\gamma}\int_f^1 \sqrt{\frac{\mu^2 - f^2}{1 - \mu^2}}\,d\mu = \pi k.$$

2. Assume s is odd. Then $\psi(\mu_0)$ - purely imaginary

$$\cos\int_f^1 |p|\,d\mu = 0,$$

or

$$\sqrt{-\gamma}\int_f^1 \sqrt{\frac{\mu^2 - f^2}{1 - \mu^2}}\,d\mu = \pi\left(k + \frac{1}{2}\right).$$

Case II - Odd resolutions of ψ. The asymptotes here will be

$$\psi \sim \sqrt{p}\,\mathrm{sh} \int_0^{\mu_0} p\, d\mu,$$

hence

$$\psi(\mu_0) \sim \sqrt{p}\,\mathrm{sh}\left[\int_0^f p\, d\mu + \int_f^1 p\, d\mu + \int_1^{\mu_0} p\, d\mu\right].$$

By separating the real and imaginary parts we have

$$\psi(\mu_0) \sim \sqrt{p}\left\{\mathrm{sh}\left[\int_0^f p\, d\mu + \int_1^{\mu_0} p\, d\mu\right] \cos \int_f^1 |p|\, d\mu -\right.$$

$$\left. - i\,\mathrm{ch}\left[\int_0^f p\, d\mu + \int_1^{\mu_0} + p\, d\mu\right] \sin \int_f^1 |p|\, d\mu\right\}.$$

1. Assuming s is even, then

$$\sin \int_f^1 |p|\, d\mu = 0,$$

or

$$\sqrt{-\gamma} \int_f^1 \sqrt{\frac{\mu^2 - f^2}{1 - \mu^2}}\, d\mu = \pi k.$$

2. Assuming s is odd, then,

$$\cos \int_{f}^{1} |p| \, d\mu = 0$$

or

$$\sqrt{-\gamma} \int_{f}^{1} \sqrt{\frac{\mu^2 - f^2}{1 - \mu^2}} \, d\mu = \pi \left(k + \frac{1}{2}\right).$$

In case II the same formula was obtained, as in case I. Thus, with even s

$$\sqrt{-\gamma} \int_{f}^{1} \sqrt{\frac{\mu^2 - f^2}{1 - \mu^2}} \, d\mu = \pi k. \qquad (3.27)$$

With odd s

$$\sqrt{-\gamma} \int_{f}^{1} \sqrt{\frac{\mu^2 - f^2}{1 - \mu^2}} \, d\mu = \pi \left(k + \frac{1}{2}\right) \qquad (3\text{-}28)$$

Formulas (3.27) and (3.28) are the analogues of "quantization law" in our case.

It should be mentioned that the formulas could be highly simplified, using the fact that at $\gamma \to \infty$, it follows that f - 1. Then

$$\frac{\mu^2 - f^2}{1 - \mu^2} = \frac{\mu + f}{1 - \mu} \, \frac{\mu - f}{1 - \mu} \approx \frac{\mu - f}{1 - \mu}, \quad f < \mu < 1.$$

By carrying out this simplification we get an integral, elementally calculable. We will have:

for even s

$$\sqrt{-\gamma}\ (1 - f) \sim 2k,$$

for Odd s

$$\sqrt{-\gamma}\ (1 - f) \sim 2k + 1,$$

or in every case

$$\sqrt{-\gamma}\ (1 - f) \sim 2k + p, \tag{3-29}$$

where p is a whole positive number, having the same parity, as s.

We get somewhat more accurate result, substituting

$$\sqrt{\frac{\mu + f}{1 + \mu}} = \sqrt{1 + \frac{f - 1}{1 + \mu}} \approx \sqrt{1 + \frac{f - 1}{f + 1}} \approx 1 + \frac{f - 1}{2\ (f + 1)}\ .$$

then

$$\sqrt{-\gamma}\ \left[1 - \frac{1 - f}{2\ (1 + f)}\right](1 - f) \sim p. \tag{3.30}$$

At $\frac{s}{f} < 0$ every reasoning remains in force, if f is substituted by f

We make a note of the following interesting circumstance. At each p, having the same parity as s, there are, as we have seen, two solutions of ψ even and odd. It does not mean, of course, that the two solutions, even and odd, correspond to the same values of parameters f, γ, since there cannot be a multiple spectrum. Here there are simply two very near eigen values, moreover, this fine structure differs only in some following terms of asymptotes, whereas, in the first term it is not visible. Fig: 3.6 shows asymptotic curves calculated from (3-30) for s = 2. Some of the values calculated on electronic computer for even solutions are shown by circles and for odd resolutions by crosses. We can actually see that each branch of asymptotes pertains to characteristic curves of even and odd type.

What does the number of zeros in fundamental function equal to? We have obtained asymptotes of eigen values, but not of fundamental functions, to be more exact, we obtained asymptotes of fundamental functions in a complex plane, but not on a real axis. Nevertheless, the number of zeros of the fundamental function on real axis could be found, by using the principle of the argument. For this we should follow the number of turns, completed by $\psi(\mu)$ with the movement of μ along the closed circuit, encompassing segment $[f, 1]$, on which can only lie all the radicals (except the radical $\mu = 0$ for odd solutions). With the movement in the top half-plane from $\mu = \mu_0$ to $\mu = 0$, or, better to say, up to certain low positive μ value, the change of argument is equal to $\sqrt{-\gamma} \int_f^{1} \sqrt{(\mu^2 - f^2)/(1-\mu^2)}\, d\mu$. as it follows from the reasoning carried out. Therefore, for even s this term, according to (3.27), is equal to π k, and for odd s it is $\pi(k + \frac{1}{2})$, according to (3.28). With total by-pass around the segment $[f, 1]$ this variation is double: it is 2 k = π p at even s and 2 π k + π = π p at odd s.

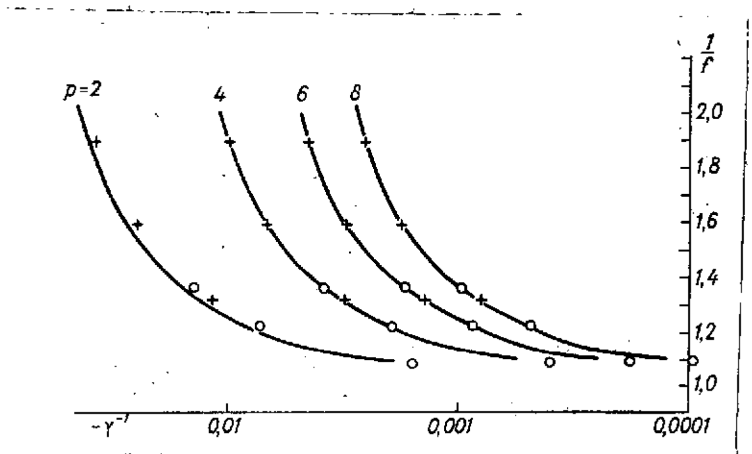

Fig: 3-6 - Asymptotes of characteristic curves at $\gamma^{-1} \to 0$.

Now we add symmetrically disposed zeros on segment $[-1, -f]$ and $\mu = 0$ for the odd solutions, and will also take into account that zeros at the ends, at $\mu = \pm 1$, add to the variation of argument quantity $p - s$ each. Hence we determine the following, irrespective of whether for even or odd s the number of zeros is p - s for even solutions and p - s + 1 odd resolutions. We denote the number of zeros by N. Let us remember denotation introduced by us for wave type modes. The fundamental functions were denoted by $\psi^{s}_{n,f}(\mu)$, where for modes of second order (and they can only be in the region of negative γ^{-1}) the index n is negative.

Comparison of values of parameter p, number n and the number of zeros N in function ψ is given in the following table:

Mode	$\psi^{s}_{-s,f}$	$\psi^{s}_{-s-1,f}$	$\psi^{s}_{-s-2,f}$	$\psi^{s}_{-s-3,f}$	. . .
Number n	-s	-s-1	-s-2	-s-3	. . .
p	s	s	s+2	s+2	. . .
Number of zeros N .	1	0	3	2	

We see, that for even solutions $n = -p - 1$, and for odd ones $n = -p$. For even $N = |n| - s - 1$, and for odd $N = |n| - s + 1$.

7. CONJUGATION OF ASYMPTOTES AT HIGH AND LOW: $^{-1}$:

We pause initially on the case $f > 0$, i.e., on the case of waves, propagating east to west. We have asymptotes at high and low γ^{-1}. In either case all modes are divided into two orders, into two bunches. The question arises, how does the asymptotes on the left at low γ^{-1} jons up with asymptotes on the right at high γ^{-1}? In particular, whether the curves of the first order on the right are the same as on the left and vice-versa? When we were speaking about the results of calculations and given corresponding curves (Fig: 2-5) we pointed out that this was not exactly like that. All these curves are of the same type, as in Fig: 3-7 (the values on axes are plotted in logarithmic scale). In every case the left portion of the curve ((-s, s) is the top curve of the first order, and the right-bottom curve of the second order. In every case also $p = 0$ for asymptotes of the second order on the left is not used. Thus, the constant p in the formula for asymptotes of the second order on the left (3-20) is bound up with wave type (n, s), $n < 0$ in the following way: $p = |n| - s$. Constant p in asymptotes of the first order (3-19) is bound up with type (n, s) $n > 0$ in this way: $p = n - s + 1$, if $p > 0$, whereas, the case of $p = 0$ pertains to mode (-s, s). Basing on results of para 5, we shall prove that this is in general conformity to law.

In para 6 of chapter 2 it was shown that at $f^2\gamma < s^2$ the number of zeros in function ξ along one mode cannot change. But in this zone lie all the curves, which are controlled on the left by asymptotes of the second order, $f^2\gamma \sim s^2/(2p + 1)^2$, except, perhaps, their end number. Thus, the curve the left portion of which is depicted by asymptotes of the second order at sufficiently high p, has an invariant throughout its extent, number of zeros in function ξ. In its left portion this number of zeros, as we know, is p, and in the right portion, where it is depicted by asymptotes of the second order on the right, $s/f \sim |n|/(|n| + 1)$, this number is $|n| - s$. Thus, $p = |n| - s$. Hence the correspondence is fixed automatically, if it is taken into account, that to each $p > 0$ for asymptotes of the second order on the left and to each $p \geqslant 0$ for asymptotes of the first order corresponds, as not quite exactly shown in para 5, one curve.

It is still not clear, whether any of the curves correspond to the case $p = 0$ for asymptotes of the second order. Now it is easy to see, that there is no such curve. On the one hand, if there were an existence of such a curve, there would have been two curves side by side: one of the first order, the other of the second, pertaining to $p = 0$. Therefore, for each of these modes ξ would have been an even function. On the other hand, the parity property of ξ is maintained along the whole mode, whereas, the asymptotes on the right show that for adjacent modes similar parity cannot exist, modes with even ξ alternate with modes of odd ξ. So modes (-s-2, s), (-s, s) and (s+1, s) correspond to even ξ, and modes (-s-3, s), (-s-1, s) and (s,s) to odd. After this the correspondence between asymptotes on the right and left is unambiguously fixed, as shown in Fig: 3-7.

It is significant, that although the invariance of ξ zeros along the mode is not proved, it is already clear, that in asymptotic zones on the right and left the number of zeros coincides. However, it is not excluded that in some in-termediate zone this number varies, and then returns to the previous value.

Let us now take the case $f < 0$. Here, the correspondence can be fixed on the basis of the following reasoning: On the right there are only the asympotes of the first order $f^2\gamma \sim n\,(n+1)$, and on the left - asympototes of the first order $f^2\sqrt{\gamma} \sim 2p + 1$ for all $p \geqslant 0$, and perhaps, one exceptional case $p = 0$ for asymptotes of the second order, i.e., $f\sqrt{\gamma} \sim -s$. Moreover, in this exceptional case the asymptotic equation is met not by ξ, but by ψ. In other words, in this case ψ, should be even, and ξ odd. Is there a curve, corresponding to these asymptotes?

It becomes clear, that this curve does exist, if it is taken into account, that the top curve (s, s), as follows from asymptotes on the right, corresponds to odd ξ, therefore, left of the top curve there cannot be a curve of the first order at $p = 0$, for which the ξ is even, and there is only our additional curve of the second order.

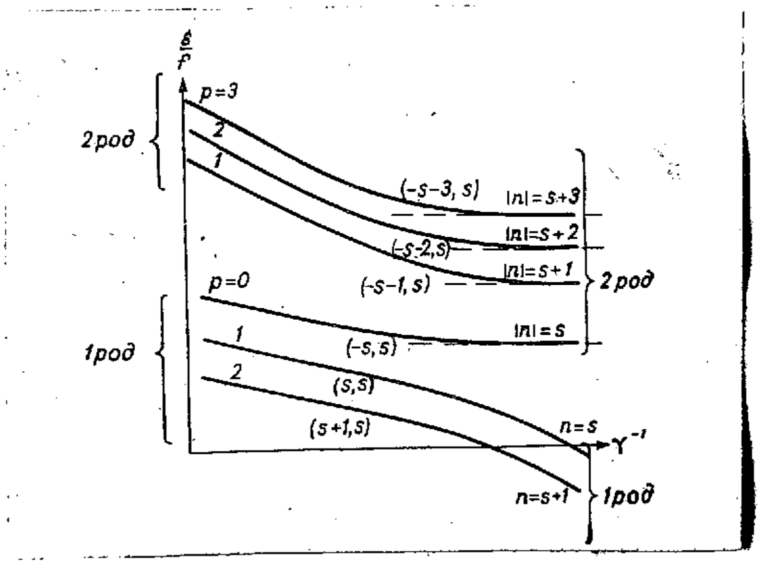

Fig: 3-7 - Conjugation of asymptotes at high and low γ^{-1}.

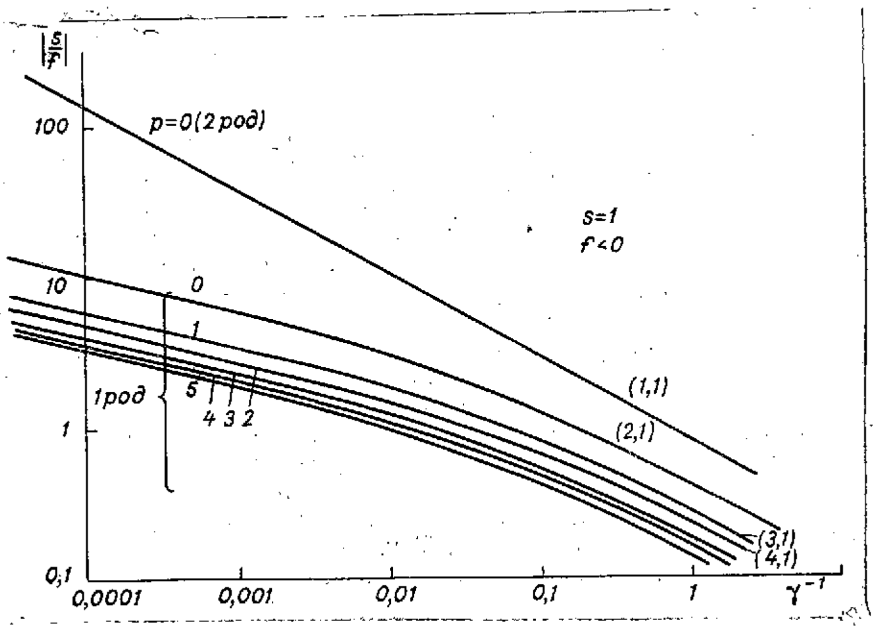

Fig: 3-8 - Characteristic curves at negative f.

Fig: 3-8 shows calculation results of characteristic curves at $f < 0$. They confirm everything that has been stated.

It should be mentioned that even in the case of $f < 0$ the number of zeros of function ξ in asymptotic zones on the right and left coincides. Here it is more difficult to notice, since $p = n - s - 1$. The number of $\xi^{s}_{n,f}$ (μ) zeros on the right is $n - s + 1$. As we have remarked in para 5, at $f < 0$ the number of zeros on the left is not p, as in the case of positive f, but $p + 2$. Hence we get the parity of the number of zeros on the right and left. As regards preserving the number of ψ zeros, nothing can be said here, since on the left investigation was of asympotes ξ, and not of ψ. We have only reminded that with positive f there is a loss of two zeros, when the f, increasing, passes through one.

CHAPTER-4.

VERTICAL STRUCTURE OF OSCILLATIONS.

1. The case of isothermal stratification.

In the two preceding chapters we have investigated equation (1.32) - one of the two equations into which the problem equation (1.30) gets split. Now let us take the second equation (1.33) for vertical component $y(x)$. This equation, in distinction from the Laplace's equation of the Theory of Tides, as has been mentioned, contains stratification chateristics of the atmosphere, entering through parameter $H(z) = R\,T(z)/g$, but does not depend either on the angular velocity of the earth's rotation ω, or on whether we consider the earth flat or spherical. Thus, we have equation

$$y'' + \left\{ -\frac{1}{4} + \frac{\sigma^2 H}{\chi g}\left(1 - \frac{\chi H}{h}\right) + \frac{H}{\chi g h}\beta \right\} y = 0, \quad (4.1)$$

where $\beta = (\chi - 1)g + \chi g\, dH/dz$ is static stability factor. The term of equation, containing, β takes into account gravitational elasticity. Boundary conditions, at which this equation should be solved, have the following appearance :

$$y' + \left(\frac{H}{h} - \frac{1}{2}\right) y = 0 \qquad \chi = 0, \quad (4.2)$$

and

$$y \text{ is limited at } \chi \longrightarrow \infty .$$

Elucidation is required, as to values of parameters σ, h for which equation (4.1) has solution satisfying conditions (4.2) and (4.3). A series of characteristic curves of the equation will appear on the plane of parameters, σ , h. Moreover, sometimes we shall study forced oscillations from a periodical force active on the surface of the earth. Boundary condition on infinity will be of the type of radiation condition.

We must remind, that we already know one class of eigenvalues, or, to say better, one characteristic curve of the equation: for any function H (z), not decreasing on infinity, there is a characteristic curve $\sigma^2 = \frac{g}{h}$. The fundamental functions in this case will be

$$y = y_o \exp \left[\int_o^x \left(\frac{1}{2} - \frac{H}{h} \right) dx \right] .$$

We named it as Peckeris solution in accordance with the name of the author, who was the first to obtain it for the case of H = const. But in chapter 1 some doubt was thrown on the physical meaning of this solution, since it results in w values, very quickly increasing with altitude.

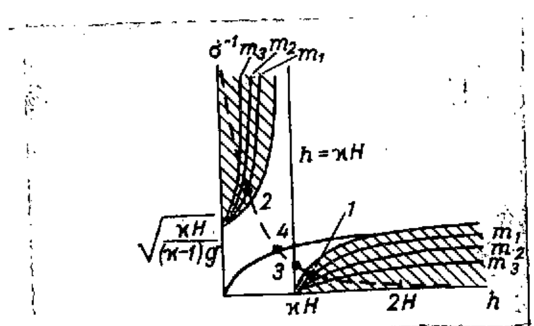

Fig. 4.1. Types of waves for flat non-rotating isothermal atmosphere.

We begin with obtaining certain specific features in disposition of characteristic curves (h), by investigating simple and many times described case of isothermal stratification (Eliassen, Kleinschmidt, 1957; Monin, Obukhov, 1958; Eckart, 1960). Thus, assuming H = const, $\beta = (\chi - 1)g$. Equation (4.1) will be

$$y'' + \left\{ -\frac{1}{4} + \frac{\sigma^2 H}{\chi g} \left(1 - \frac{\chi H}{h}\right) + \frac{\chi - 1}{\chi} \frac{H}{h} \right\} y = 0. \tag{4.4}$$

Its factors are now constant. For those σ and h values, for which the term in braces is positive, all resolutions are of oscillation type and are limited at infinity. Apparently, it is always possible to compose linear combination of linearly independent resolutions so, as to meet the single boundary condition on the earth's surface. Thus, any such pair of σ and h is eigenvalue and the problem has a continuous spectrum.

So, the continuous spectrum consists of all the , h points, located within the zone

$$-\frac{1}{4} + \frac{\sigma^2 H}{\chi g} \left(1 - \frac{\chi H}{h}\right) + \frac{\chi - 1}{h} \frac{H}{h} > 0. \tag{4.5}$$

As before we shall use for illustration the plane σ, h. In Fig. 4.1 the zone (4.5) is hatchured. Boundary of the zone, hyperbola

$$-\frac{1}{4} + \frac{\sigma^2 H}{g} \left(1 - \frac{\varkappa H}{h} \right) + \frac{\varkappa - 1}{\varkappa} \frac{H}{h} = 0,$$

has a horizontal asymptote $\sigma^2 = \varkappa\, g/4H$ and a vertical asymptote $h = 4\ (\varkappa - 1)H/$. The intersection point of hyperbola with the axis of abscissae is $h =$ H, and with axis of ordinates $\sigma^{-1} = \varkappa\, h/(\varkappa - 1)g$. The hyperbola is located in respect of its asymptotes as shown in the figure, because always $\varkappa > 4(\varkappa - 1)/\varkappa$.

Besides, the indicated continuous spectrum there is also a discrete spectrum. The latter could be obtained in the following way. If σ and h are such that the term in braces in equation (4.4) is negative, then one of the solutions has the appearance of a vanishing exponent, and the others - of growing. Boundary condition at infinity is met only by the vanishing exponent. The resolution, therefore, should be of the type e^{-mx} . Boundary condition (4.2) and equation (4.4) give

$$m = \frac{H}{h} - \frac{1}{2},$$

$$m^2 = \frac{1}{4} - \frac{\sigma^2 H}{\varkappa g} \left(1 - \frac{\varkappa H}{h}\right) - \frac{\varkappa - 1}{\varkappa} \frac{H}{h},$$

or

$$\left(\frac{H}{h} - \frac{1}{2} \right)^2 = \frac{1}{4} - \frac{\sigma^2 H}{\varkappa g} \left(1 - \frac{\varkappa H}{h}\right) - \frac{\varkappa - 1}{\varkappa} \frac{H}{h}, \qquad (4.6)$$

in which case $m > 0$, i.e.

$$\frac{H}{h} - \frac{1}{2} > 0.$$

Equation (4.6) breaks up into two

$$h = \varkappa H$$

and

$$\sigma^2 = \frac{g}{h}$$

at $h < 2H$. The second of these curves as the Peckeris curve. The first curve corresponds to the so called two-dimensional waves. In oscillations of this type the vertical velocity is equal to zero not only on the hard surface, but also identically. Amplitude of these oscillations y (x) does not change the sign. Fig. 4.1 shows these two curves of discrete spectrum: for two-dimentional waves this is a vertical straight $h = \varkappa H$, and for Peckeris curves - parabola, touching upon the boundary of continuous spectrum zone at $h = 2H$.

In spite of the fact that the points of continuous spectrum fill up the entire zone, it is quite easy to imagine this zone as consisting of continuum of individual characteristic curves. This could be done in the following way. The resolutions of continuous spectrum have the form

$$y = a \sin mx + b \cos mx,$$

where

$$m^2 = -\frac{1}{4} + \frac{\sigma^2 H}{\varkappa g}\left(1 - \frac{\varkappa H}{h}\right) + \frac{\varkappa - 1}{\varkappa}\,\frac{H}{h}. \tag{4.7}$$

Thus, m means vertical wave number. With the fixed vertical wave number (4.7) m converts into equation of curve, which we shall name the characteristic curve with preset vertical wave number. Fig.4.1 shows several of these curves for various m values, in the figure

$m_1 < m_2 < m_3$. At $m = 0$ the curve gets converted into boundary of the continuous spectrum zone. Each curve consists of two branches. One of them emerges from point $h = \varkappa H$ on the axis of abscissae and has a horizontal asymptote $\sigma^{-1} = \sqrt{H/\varkappa g \left(m^2 + \frac{1}{4} \right)}$. The other emerges from point $\sigma^{-1} = \sqrt{H/(\varkappa - 1)g}$ on the axis of ordinates and has a vertical asymptote $h = (\varkappa - 1) H/\varkappa \ (m^2 + \frac{1}{4})$.

2. Acoustic and gravitational waves.

In order to give physical interpretation to obtained results, we should remember, that eigen values of the problem are obtained at the intersection of the characteristic curves of the equation for horizontal portion of the solution, i.e. the Laplace's tidal equation, with characteristic curves of equation for the vertical part of the resolution, i.e. in this case equation (4.4). Let us take, at first, an absolutely simple case - model of flat non-rotatory earth. Then, as we saw, instead of the Laplace's equation of the Theory of Tides there is an ordinary Laplace's equation on a plene, at which the spectrum is continuous and to each horizontal wave number k corresponds to a characteristic curve (2.25), i.e.,

$$\sigma = \sqrt{gh}\, k.$$

In Fig. 4.1 one of these curves is shown by a dotted line. The same figure shows several curves of continuous specturm in equation (4.4), and also two curves of discrete spectrum for two-dimensional waves and Peckeris curve. The numerals mark various intersection points of these curves. We start from points 1 and 2, intersection points of the " horizontal equation " characteristic curves $\sigma = \sqrt{gh}\, k$ with curve (4.7). From the two equations it is possible to exclude h and to obtain frequency σ as a function of two wave numbers k and m. For σ^2 we obtain

$$\frac{\sigma^2 H}{\varkappa g} + \frac{\varkappa - 1}{\varkappa} \frac{gHk^2}{\sigma 2} = m^2 + \frac{1}{4} + H^2k^2, \qquad (4.8)$$

the same equation as in Monin's and Obukhov's article (1958). This quadratic in relation to σ^2 equation has two solutions, corresponding to points 1 and 2. Point 1 corresponds to acoustical waves, and point 2 - to gravitational waves (below will be explained the meaning of this denotation). Thus, the bunch of curves in Fig. 4.1 emerging from point $h = \varkappa H$ on the axis of abscissae, determines, on intersection with characteristic curves of horizontal equation, solutions, corresponding to acoustical waves. The bunch of curves, emerging from point $\sigma^{-1} = \sqrt{\varkappa H/(\varkappa -1)g}$ on axis of ordinates, corresponds to gravitational waves. All gravitational frequencies are lower than frequency $\sigma = \sqrt{(\varkappa -1)g/\varkappa H}$, named Brent-Weisel frequency. For H = 8 km it corresponds to periods higher than about 330 sec. All acoustical frequencies are higher than $\sqrt{\varkappa g/4H}$, which corresponds to periods less than 300 sec. Moreover, the dynamically equivalent depths h for acoustical waves are greater than $\varkappa$ H, i.e., about 11.2 km, and for gravitational waves they are less than $4(\varkappa -1)H/$, i.e., 9.1 km. Correspondingly, phase velocities $\sqrt{gh}$ of acoustical waves are over 330 m/sec., and of gravitational waves less than 298 m/sec.

Next we turn to point 3. It corresponds, as has been stated, to two-dimensional waves in the sense, that these have no vertical velocity and at every altitude of oscillation are in one phase. In contrast, the acoustical and gravitational waves are internal, which, besides the horizontal wave number k, have also the vertical number m. Waves, pertaining to point 3, are know as Lamb's waves.

Finally point 4 corresponds to Peckeris resolution. It lies on parabola $\sigma^2 = g/h$ and on the characteristic curve of horizontal equation $= \sqrt{gh}\,k$. If h is eliminated from here, there will be the following bond between frequency and the horizontal wave number:

$$\sigma^2 = gk. \tag{4.9}$$

It is exactly in this form the term was written for frequency in Peckeris article (1948). Now we shall explain the meaning of the names " acoustical " and " gravitational " waves. They can be, for instance, distinguished by their behavior in two ultimate cases : at $\varkappa \longrightarrow \infty$ and $\varkappa \longrightarrow 1$. The first ultimate transition can be interpreted as transition to incompressibility. In fact, we are investigating polytropic processes, in which the ln p $^{-\varkappa}$, or

$$\frac{1}{\varkappa} \ln p\rho^{-\varkappa} = \frac{1}{\varkappa} \ln p - \ln \quad .$$

are preserved.

At $\varkappa \to \infty$ hence follows ρ preservation as constant, i.e. incompressibility. The second limited transition, at $\varkappa \longrightarrow 1$, corresponds to indifferent equilibrium of atmosphere, when the particle, isothermally displaced vertically ($\varkappa = 1$), has the same temperature, as the surrounding particles, and does not experience any expelling force from either side. The static stability factor $\beta = (\varkappa - 1)g$ converts into zero. And what happens during these two ultimate transitions with acoustical and gravitational frequencies? At $\varkappa \to \infty$, i.e., with transition to incompressibility, the " acoustical " bunch of curves, emerging from point $h = \varkappa H$, withdraws into infinity. With this the intersection point of characteristic curves of equations for horizontal and vertical components at some fixed k and m withdraws into infinity, simultaneously approaching the axis of abscissae. Thus, the phase velocities of these waves strive to endlessness, and their periods to zero. As regards the frequencies of the second " gravitational " set, they change only very negligibly. In particular, there is a slight increase of their limiting velocity, the Brent-Weisel velocity; it becomes equivalent to $\sqrt{g/H}$ instead of $\sqrt{(\varkappa - 1)g/\varkappa H}$. Thus, for acoustical waves, in contrast to gravitational, the compressibility is found to be the decisive factor.

On the contrary, at $\varkappa$ 1, the acoustical frequencies vary insignificantly, whereas the entire gravitational bunch of characteristic curves in Fig. 4.1 rises into infinity, as the Brent-Weisel frequency $\sqrt{(\varkappa - 1)g/\varkappa H}$ converts into zero. Thus, all the gravitational frequencies get converted into zero and the periodical oscillating process becomes transformed into a stationary whirling motion. The determining cause of these oscillations could be taken the stable stratification, or Archimedean buoyancy. With transition to indifferent equilibrium the cause of these oscillations disappears and they are replaced by stationary motions.

The reasons stated above justify the names given. In chapter 5 these reasons will be confirmed by additional arguments, pertaining to structure and energy composition of oscillations.

3. Gyroscopic-inertia waves.

Let us turn now to a more composite case of rotating spherical atmosphere. The corresponding curves of Laplace's tidal equation were described in preceding chapters. Fig. 4.2 shows how characteristic curves of Laplac's tidal equation intersect with characteristic curves of vertical equation (4.4). For a curve of the first order asymptotically approaching with h increase the axis of abscissae, the situation changes very little in comparison with that, which took place in the model of flat non-rotating earth. The only difference consists in the fact, that instead of the continuum of the horizontal equation characteristic curves we have a discrete, though quite dense, set of them and the formula for characteristic curve $\sigma = \sqrt{gh}\, k$ is not exact any more, but only asymptotically true at sufficiently high h values.

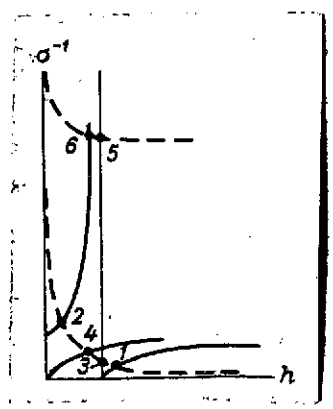

Fig. 4.2. Types of waves for spherical rotating isothermal atmosphere.

The high-frequency gravitational and two-dimensional waves, and also the acoustical waves depend very little on such factors as the shape of the earth and angular velocity of its rotation, which can only affect waves of planetary scale.

But in Fig. 4.4. we see intersection points of charateristic curves of eqution (4.4) with characteristic curves of the second order of Laplace's tidal equation, which were not present before. These intersection points are marked by numerals 5 and 6. The maximum frequencies for these waves are given by Rossbi-Haurwitz formulas $\sigma = 2\omega s/n(n+1)$, since the curves of second order are nowhere below their asymptotes, determinable by this formula. Thus, here we are speaking of extremely low-frequency oscillations, with the least period equal to a day, but, as a rule, considerably higher. If the angular velocity of rotation is assumed to be zero, the horizontal asymptotes of characteristic curves rise to infinity, i.e. the periods become infinitely long, and instead of waves we have stationary motions. Thus, the physical cause of these oscillations is the rotation of the atmosphere, gyroscopic rigidity.

Here distinction should also be made of oscillations, corresponding to points 5 and 6. The first belong to discrete spectrum of vertical equation. Their amplitude y(x) vanishes with altitude, and they have no vertical velocity. Their frequencies are approximately $\sigma \approx 2\omega s/n(n+1)$. If more exact asymptotes are used of the characteristic curves of Laplace's tidal equation, for instance, (2.18), and to substitute there instead of h (entering through y) the term $h = \chi H$, we shall get

$$\frac{2\omega s}{\sigma} = n(n+1)\,\frac{4a^2\omega^2}{\varkappa_g H}\left[\frac{(n-s)(n+s)(n+1)^2}{(2n-1)(2n+1)n^2} + \right.$$

$$\left. + \frac{(n+1-s)(n+1+s)\,n^2}{(2n+1)(2n+3)(n+1)^2}\right]. \qquad (4.10)$$

These two-dimensional gyroscopic-inertia waves are knows as Rossbi waves.

As regards oscillations, corresponding to point 6, they are first of all internal, since their vertical wave number is distinct from zero. Their existence is bound up with combination of two effects: gyroscopic effect caused by the earth's rotation, and the effect of temperature stratification. The frequencies of these oscillations get converted into zero with conversion into zero of ω , and with transition to indifferent stratification ($\varkappa = 1$). Therefore, they are the internal gravitational - gyroscopic waves. In order to obtain a formula for frequencies of these oscillations, it is necessary to take a formula for characteristic curves of vertical equation (4.7) and to eliminate from them h. Since the intersection of characterisitc curves occurs within the zone of extremely low frequencies, it is possible to assume in (4.7) $\sigma = 0$. Then we shall have

$$\frac{2\omega s}{\sigma} = n(n+1) + \frac{(4m^2+1)\,a^2\omega^2\varkappa}{(\varkappa-1)gH}\ \times$$

$$\times\left[\frac{(n-s)(n+s)(n+1)^2}{(2n-1)(2n+1)n^2} + \frac{(n+1-s)(n+1+s)n^2}{(2n+1)(2n+3)(n+1)^2}\right]. \qquad (4.11)$$

This formula is asymptotic. It gives good results for not too high m numbers. For oscillations with low horizontal wave numbers n, s and high vertical wave numbers m this formula is not suitable. In the last case it is possible to use asymptotes, obtained in chapter 3 for low γ^{-1}. Formulas (4.10) and (4.11) were given in the Author's article (1961). In Hough's (1898) and Yaglom's (1953) works formulas are given for frequencies of two-dimensional waves, where instead of asymptotes (2.18) use was made of asymptotes (2.21). The internal waves were not analyzed by Hough and Yaglom.

A few words regarding difference between waves propagating west to east and east to west. It is known that the difference between these waves is expressed in the f sign. It has been shown above that the characteristic curves of Laplace's tidal equation in the case $f < 0$, i.e., for waves, directed west to east, had no horizontal asymptotes. In this case there are no gyroscopic waves. In other words, all gyroscopic waves, both external and internal, always propagate east to west.

As regards the quick waves (acoustical, quick gravitational and Lamb's waves) their frequencies are such that they are determined very precisely by asymptotes (2.6) $|f| \sim \sqrt{n(n+1)\gamma}$, completely independent of f sign. Asymmetry in direction begins to be felt only for very slow gravitational waves, where the f sign enters into the next term of asymptotes. [See (2.19) - asymptotes at high γ^{-1} and (3.19) - asymptotes at low γ^{-1}].

The next remark we make in relation to negative h values. We know that in the left semiplane, at negative h values, lie some characteristic curves of Laplace's tidal equation. Moreover, these curves are wholly in the zone $f \leqslant 1$, i.e., within the zone of periods higher than half a day. As regards the curves (4.7), they also penetrate into left semiplane, into the zone of negative h. But this occurs at $\sigma > \sqrt{(\varkappa-1)g/\varkappa H}$, i.e., if the periods are less than 5 min. Therefore, the characteristic curves of Laplace's tidal equation and of equation for vertical component cannot intersect in the negative semiplane. Thus, for natural oscillations h is always positive.

And one more remark. As can be seen from the figure, at sufficiently low σ, the characteristic curves are very near to their vertical asymptotes. It means, that in the estimate of these natural oscillations in equation (4.1) it is possible to disregard the terms containing σ. If we analyze the process of obtaining this equation, paying attention to whence these terms are obtained, we will discover, that these terms originate for the left portion of Euler's third equation (1.23), i.e., with estimate of vertical acceleration. Disregarding these terms is general in meteorology, which studies the slowest processes, and it is known as approximations of quasistatics.

In what way will the pattern of natural oscillations change in quasistatic approximation? All the curves, corresponding to gravitational waves, will be substituted by their vertical asymptotes. The one which is not substituted will remain straight $h = \varkappa H$, since it is vertical. The acoustical bunch, as well as Peckeris curve, disappears.

Hence it is possible to come to the conclusion, that the quasistatic approximation does not change the Lamb's and Rossbi waves, so important in meteorology, insignificantly distorts the slow gravitational waves with periods higher than about 15-20 min., highly distorts the quick gravitational waves and completely destroys the set of acoustical waves.

4. The case of real stratification.

As mentioned in the introduction, we are taking as temperature profile the standard atmosphere CIRA 1961 (see Fig. 1.1). Starting from a certain altitude the temperature rises almost linearly. We shall assume further rise of temperature as linear. It should be taken into account in this case, that our equations generally have physical meaning only upto an altitude of about 150 km, therefore, the style of temperature profile above this altitude is of no significance. Only those properties of resolutions have physical meaning, which are not highly dependent on the behavior of equation factors at high altitude.

Due to temperature rise with altitude (it begins at an altitude of 90 - 100 km; the region above this level is known as thermosphere) all solutions of our equations, except one (for σ and h data), are found to be quickly rising with altitude; one solution, on the contrary, just as quickly decreases exponentially. Thus, the boundary condition, set at infinity, picks out one out of all the solutions. However, uptil now, not for every of such solution, i.e., not for every pair of σ , h, the condition is also met on the surface of the earth. Those values of parameters, for which it is fulfilled, are the eigen values. Therefore, in the present case the spectrum is always discrete. We will note first, certain general regularity in disposition of characteristic curves on the σ , h plane, then describe the method of their calculations and the results. Finally, it will be found that these curves are very similar to curves for isothermal atmosphere.

First of all, a few words should be said regarding negative h values. Here the position is very different from that in the case of isothermal atmosphere. With high values of vertical coordinate equation (4.1) could approximately be written as

$$y'' - \frac{\sigma^2 H^2}{gh} y = 0.$$

At negative h both the linearly independent solutions of this equation are of oscillating type and limited at infinity. This shows, that at negative h there is a continuous spectrum, and any pair of σ , h belongs to this spectrum, in contrast to what was for $h > 0$. Formally, we are obliged to analyse these solutions also. However, they could hardly have any physical meaning. This is clear from the following : The characteristic curves of the Laplace's tidal equation fall into the region of negative h values only at very low frequencies, $\sigma < 2\omega$. But for such σ values the term $\sigma^2 H^2/gh$, which determines the asymptotes of fundamental functions, begins to prevail over the term $H\beta/\varkappa\, gh$ only at very high χ values, where

$$\frac{\beta}{\varkappa H} < \sigma^2 < 4\omega^2 .$$

Below this altitude the term in braces in (4.1) is negative, the solution very quickly rises, since the sign of the second derivative of the solution coincides with the sign of the solution itself, and at $h < 0$, according to the boundary condition (4.2), the same sign has also the first derivative. Thus, the solution rises very quickly at least up to altitude, where $4\omega^2 > \beta / \varkappa H$, i.e.,

$$H > \frac{\beta}{4\omega^2 \varkappa} .$$

If we consider that β has the order 10 m/sec^2., then this evaluation gives for H values of about 10000 km, or for temperature hundreds of thousands of degrees. Only above this altitude the solution begins to decrease, so that finally the boundary condition at infinity is found to be fulfilled. Hence it is possible to come to the conclusion that the solution is really distinct from zero only at very high altitudes, for which our equation is written purely formally and does not reflect any physical reality. Thus, these solutions appear due to conventionality of the mathematical model.

We may also add, that, by adopting approximation of quasistatics, we would have

$$y'' = \left(\frac{1}{4} - \frac{H\beta}{gh} \right) y.$$

At $h < 0$ the term in round brackets is positive, i.e., y'' and y have similar signs. If it is also taken into account that due to boundary condition $y' = (\frac{1}{2} - \frac{H}{h})y$ at $\chi = 0$ similar signs will have y' and y, and it will become clear, that in this case the resolution is a monotonically rising function of altitude and will never die out.

In the investigation of natural oscillations there is no need to consider the negative h values. The forced oscillations are a different matter. In their investigation special attention is being paid lately to the region of negative h.

When the characteristic curves are calculated on electronic computer it is impossible to solve the equation in endless region. The solution has to be broken off at a certain level, substituting for the remaining portion of the region some imaginary boundary conditions. We have fixed the top limit at an altitude of 200 km, assuming, that the solution derivative there is equal to zero. Another natural version of imaginary boundary condition at an altitude of 200 km, is to assume there, as on a hard surface, vertical velocity equal to zero, i.e., to take condition (4.2) at the top limit, the same as on the bottom.

5. Calculation of characteristic curves.

How to calculate characteristic curves of equation (4.1)? We will be, in a way, probing the whole (σ, h) plane along some test curves in search for a pair of eigen values , h. As test curves we take the already familiar curves, depicted by equation

$$\sigma = \sqrt{gh}\ k,$$

which are the characteristic curves of horizontal equation in the case of model of the flat non-rotatory earth, k being horizontal wave number. We substitute σ^2 from this equation into equation (4.1), thereafter obtaining an equation with one parameter h

$$- y'' + \left(\frac{1}{4} + k^2H^2 \right) y = \frac{H}{\varkappa g} \left(gk^2h + \frac{\beta}{h} \right) y. \qquad (4.13)$$

Let us take solution of this equation, which meets at infinity the limiting condition being zero at infinity (or meeting at altitude x_o, corresponding to z = 200 km, the adopted imaginary boundary condition). We denote this solution by φ (x, h). We make up function

$$M\ (h) = \frac{\frac{d\varphi(0,h)}{d}}{\varphi\ (0,h)}\ . \qquad (4.14)$$

Integrating equation (4.13) by means of some numerical method from point $x = x_o$ to $x = 0$, we can calculate for each value of h the value of function M(h) and plot its curve. At the same time it is possible to plot curve of function

$$N\ (h) = \frac{1}{2} - \frac{H\ (0)}{h}\ . \qquad (4.15)$$

Those h values, at which these curves intersect, are sought for eigen values. In fact, the solution of equation (4.1) φ (x , h) then meets both the condition at level x_o(in plotting), and the condition on the surface of the earth since

$$\frac{d\varphi(0,h)}{d} = \left(\frac{1}{2} - \frac{H(0)}{h}\right)\varphi(0,h).$$

Function N(h) is monotonously rising. What genèral reasons could be given in relation to function M (h)? Let us calculate, for instance, derivative dM/dh

$$\frac{dM}{dh} = \frac{1}{\varphi^2(0,h)}\left[\frac{d\varphi'(0,h)}{dh}\varphi(0,h) - \varphi'(0,h)\frac{d\varphi(0,h)}{dh}\right]$$

(the prime mean derivatives from x). Function φ (x,h) meets the equation

$$-\varphi'' + \left(\frac{1}{4} + k^2H^2\right)\varphi = \frac{H}{\chi g}\left(k^2gh + \frac{\beta}{h}\right)\varphi. \qquad (4.16)$$

Let us differentiate this equation by h

$$-\frac{d\varphi''}{dh} + \left(\frac{1}{4} + k^2H^2\right)\frac{d\varphi}{dh} = \frac{H}{\chi g}\left(k^2gh + \frac{\beta}{h}\right)\frac{d\varphi}{dh} +$$

$$+\frac{H}{\chi g}\left(k^2g - \frac{\beta}{h^2}\right)\varphi \quad . \qquad (4.17)$$

We multiply equation (4.16) by d /dh and substract from it equation (4.17), multiplied by φ ;, then integrate the difference from 0 to x_o . We get

$$\varphi'(0,h)\frac{d\varphi(0,h)}{dh} - \varphi(0,h)\frac{d\varphi'(0,h)}{dh} =$$

$$= - \int_0^\infty \frac{H}{\chi g} \left(k^2 g - \frac{\beta}{h^2} \right) \varphi^2 \, dx.$$

This term we substitute in formula for derivative dM/dh

$$\frac{dM}{dh} = \frac{1}{\varphi^2(0,h)} \int_0^\infty \frac{H}{\chi g} \left(k^2 g - \frac{\beta}{h^2}\right) \varphi^2(x,h) dx.$$

Thus, the sign of derivative depends on the sign of integral $\int_0^\infty H \left(k^2 g - \frac{\beta}{h^2}\right) \varphi^2(x,h)\, dx$. Now we take into account that $\beta(x)$ is limited both from the top and from the bottom

$$\beta_1 \leq \beta \leq \beta_2 \tag{4.18}$$

(approximately it may be assumed, that $\beta_1 \approx 1.4$ m/sec^2., $\beta_2 = 12$ m/sec^2. Hence it follows, that at $h > \sqrt{\beta_2/k^2 g}$ we have $dM/dh > 0$, and at $h < \sqrt{\beta_1/k^2 g}$ - $dM/dh < 0$.

The fact, that $dM/dh > 0$ at sufficiently high h does not mean monotonous rising of function M (h). There is a calculated multiple of the break points of function M(h), these h values are the eigen values of equation (4.13) with a simpler boundary condition at x = 0, namely, y(0) = 0. Between these break points the M(h) rises from $-\infty$ to ∞.

In the region of h values, where $dM/dh < 0$, there is also a calculated multiple of break points, accumulating towards h = 0, and between each two break points M(h) decreases from $+\infty$ to $-\infty$. The typical curve of the function in question is shown in Fig. 4.3 for $k = 0.2\ km^{-1}$.

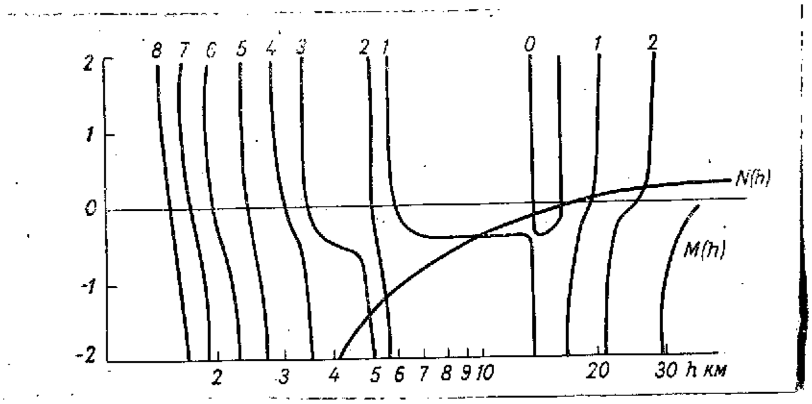

Fig. 4.3. Functions M(h) and N(h).

The above reasoning could be made somewhat more precise. Its short coming was the existence of zone $\sqrt{\beta_1 / k^2 g} < h < \sqrt{\beta_2 / k^2 g}$, in which the sign dM/dh remained undetermined. This could be eliminated, if we take a more complex function

$$P(h) = \frac{M(h) - N(h)}{\sigma^2 - \frac{g}{h}} = \left[M(h) - N(h)\right] \frac{h}{g(k^2h^2 - 1)} .$$

Apparently, the eigen values of the problem are obtainable as the zeros of this function. Let us calculate its derivative:

$$\frac{dP(h)}{dh} = \left(\frac{dM}{dh} - \frac{H(0)}{h^2}\right) \frac{h}{g(k^2h^2 - 1)} - (M - N) \frac{k^2h^2+1}{g(k^2h^2 - 1)^2} .$$

or

$$\frac{dP(h)}{dh} = \left[\frac{1}{\varphi^2(0,h)} \int_0^\infty \frac{H}{\chi g}\left(k^2g - \frac{\beta}{h^2}\right)\varphi^2 dx - \frac{H(0)}{h^2}\right] \quad X$$

$$X \frac{h}{g(k^2h^2 - 1)} - (M - N) \frac{k^2h^2 + 1}{g(k^2h^2 - 1)^2} .$$

Now we shall convert the integral. As will be shown in the next chapter, this integral is closely bound up with the concept of energy. The following identity occurs, which could be proved by partial integration:

$$\int_0^\infty \frac{x}{h}\left[\sigma^2 H\varphi + g(\varphi' - \frac{\varphi}{2})\right]^2 dx + \int_0^\infty xg\sigma^2\left[\varphi' + (\frac{H}{h} - \frac{1}{2})\varphi\right]^2 dx =$$

$$= \left(\sigma^2 - \frac{g}{h}\right)\int_0^\infty H\left(\frac{2}{\sigma} - \frac{\beta}{h}\right)\varphi^2 dx - xg\left\{ \left(\sigma^2 + \frac{g}{h}\right)\left[\varphi'(0) - N\varphi(0)\right] + \right.$$

$$\left. + \left(\frac{H(0)}{h}\right)\left(\sigma^2 - \frac{g}{h}\right)\varphi(0)\right\} \varphi \quad (0).$$

The left portion of this identity is positive. We denote it by E. Considering that φ (0) = Mφ (0), we get

$$\int_0^\infty \frac{H}{\chi g}\left(k^2 g - \frac{\beta}{h^2}\right)\varphi^2 dx = \frac{E}{(k^2h^2 - 1)\, xg^2} +$$

$$+\left[\frac{k^2h^2 + 1}{h(k^2h^2 - 1)}(M - N) + \frac{H}{h^2}\right]\varphi^2(0).$$

Finally, substituting this term into formula for dP/dh, we shall have

$$\frac{dP\ (h)}{dh} = \frac{Eh}{(k^2h^2 - 1)^2\ xg^3\varphi^2\ (\ 0,h\ )} > 0.$$

In this way there is no zone of undetermined sign. The zeros of P(h) function (i.e., the eigen values of our problem) alternate with break points (i.e., eigen values at zero condition on earth's surface), if we do not count the additional eigen value $\sigma^2 = g/h$. In other words, Fig.4.3 shows relative position of M(h) and N(h) curves in the general case. There is only one branch of M(h) curve, both ends of which withdraw into infinity of one sign, the branch, on which lies the point $\sigma^2 = g/h$ (or $k^2\ h^2 - 1 = 0$).

Fig. 4.3 also shows the curve N(h) and the intersection points of both the curves. This is the sought for eigen value h. The obtained h and σ values are plotted on plane (h, σ^{-1}). Then the k somewhat changes, i.e., the test curve shifts, and the whole procedure is repeated. As a result there is a combination of all the characteristic curves on plane (h, σ^{-1}).

Fig. 4.4 shows the main calculation result of equation (4.1) characteristic curves for vertical component. The first and the basic thing that can be said by looking at the figure is that the general qualitative pattern of characteristic curve position in a plane is very similar to the pattern, which took place for the isothermal atmosphere. The curves are also mainly grouped into two bunches, of which one corresponds to oscillations with quite high periods, and the other, in contrast, to quick, short-period oscillations. It is natural, as previously, to call the first oscillations gravitational, and the second-acoustical. The resemblance consists also in the existence in the present case of a curve, very similar to two-dimensional resolution $h = \varkappa H$ for isothermal atmosphere. Of course, now this vertical straight $h \approx 10$ km is complex : it consists of individual portions of characteristic curves, replacing one another in sequence. We shall denote this straight (slightly curved at the bottom) as the main complex mode. In the next chapter we shall see that even the properties of corresponding solutions are very similar to properties of the isothermal atmosphere two-dimensional solutions.

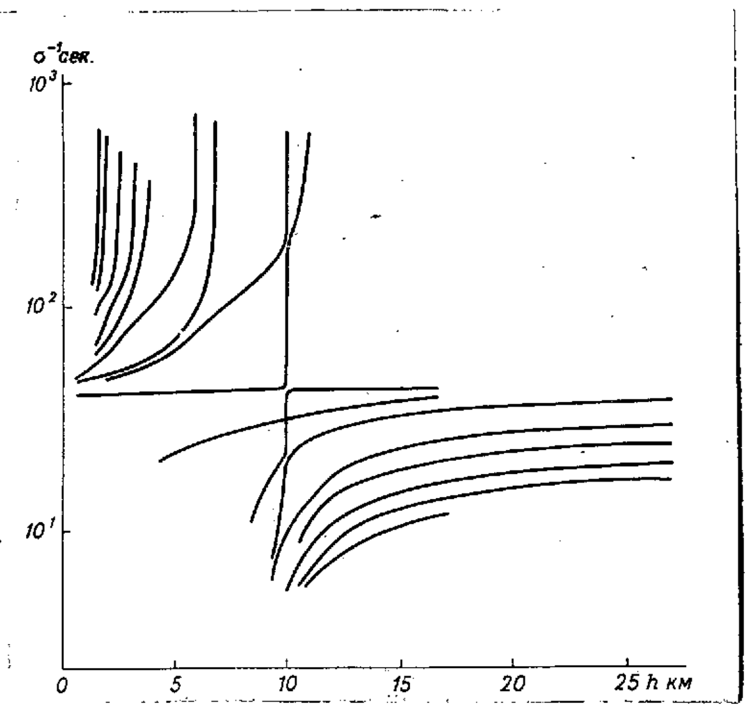

Fig. 4.4. Characteristic curves of equation (4.1) in the case of real stratification.

6. Short-wave asymptotes.

Even in the figure it can be seen, that the bunches still have the tendency to gather at points on axes, as it was in the case of isothermal atmosphere. This can be proved exactly, and these points found. We shall find, for instance, asymptotes of acoustical waves at $\sigma \longrightarrow \infty$. If we still search for intersection points of characteristic curves with " test " curves $\sigma = \sqrt{ghk}$, then $\sigma \longrightarrow \infty$ will correspond to $k \longrightarrow \infty$, i.e., to asymptotes of short waves. The h eigen values, i.e., the intersection points of M(h) and N(h) curves alternate with M(h) break points, i.e., with eigen values of a simpler marginal problem at boundary conditions on the earth's surface $y(0) = 0$. In other words, the characteristic curves of one marginal problem alternate with characteristic curves of another. If we prove, that characteristic curves of a simpler problem gather at one point on the axis of abscissae, the same will be proved also for the characteristic curves of the main problem.

Substituting in equation (4.1) the variable according to formula $x = \sqrt{h\xi}$, we get the marginal problem

$$- y'' + \left(\frac{\sigma^2 H^2}{g} - \frac{H\beta}{xg}\right)y = \left(-\frac{1}{4} + \frac{\sigma^2 H}{xg}\right)hy, \quad y(0)=0. \qquad (4.19)$$

It is required to investigate the behavior of eigen value h at $\sigma \longrightarrow \infty$. This is also a problem of quasiclassical type, similar to the one solved in chapter 3. First of all it should be mentioned, that in each of the round brackets in (4.19) may be left only one term, containing σ^2, as we are studying the behavior of solutions at high σ^2. Now we have

$$- y'' + \frac{\sigma^2}{xg} H(xH - h)y = 0.$$

This equation is already similar to Schredinger's equation and the asymptotes are given by Bohr's quantization

$$\frac{\sigma}{\sqrt{xg}} \int_{\xi_1}^{\xi_2} \sqrt{H(\sqrt{h}\xi)\left[h - xH \sqrt{h}\xi\right]} d\xi = (n + \frac{1}{2})\pi ,$$

where is a whole number, the number of eigen value, and ξ_1, ξ_2, turning points at which $h = \varkappa H$. Returning to variable x, we shall have

$$\frac{\sigma}{\sqrt{xgh}} \int_{x_1}^{x_2} \sqrt{H(x)\left[h - xH(x)\right]} dx = \left(n + \frac{1}{2}\right)\pi , \qquad (4.20)$$

where x_1, x_2 are turning points, at which $h = \varkappa H$. At $\sigma \to \infty$ the integral should be striving to zero, i.e., all the " levels " drop down to the bottom of " potential pit " H, or

$$\lim_{\sigma \to \infty} h_n = xH_{min}. \qquad (4.21)$$

Thus, it is proved, that all the characteristic curves of acoustical type in the limit case of high frequencies, or short waves, gather at point $h = \varkappa H_{min}$ on the axis of abscissae. Moreover, as in chapter 3, the conclusion may be drawn here that the solution y is mainly concentrated in the region, between the turning points $x_1 < x < x_2$, and outside this region it quickly vanishes. In other words, the high-frequency solutions are concentrated in a narrow layer of minimum H values (i.e., minimum temperatures) around altitude 84 km (slightly less high-frequency waves may concentrate also in the zone of the second minimum in the vicinity of altitude 17 km). We shall speak of this later on.

Quite similarly the investigation is made of the limiting case of short gravitational waves. This requires finding solution asymptotes of equation (4.1) at $h \longrightarrow 0$. Again, discarding low terms, we will have

$$- y'' + \frac{H^2}{gh}\left(\sigma^2 - \frac{\beta}{xH}\right) y = 0,$$

$$y(0) = 0.$$

Asymptote at $h \longrightarrow 0$ of quantization type has now the following look:

$$\frac{1}{\sqrt{gh}} \int_{x_1}^{x_2} H \sqrt{\frac{\beta}{xH} - \sigma^2}\, dx = \left(n + \frac{1}{2}\right)\pi . \qquad (4.22)$$

Hence it follows, that

$$\lim_{h \to 0} \sigma^2 = \left(\frac{\beta}{xH}\right)_{max} . \qquad (4.23)$$

Thus, frequency of the very short gravitational waves is similar to maximum Brent-Wysel frequency $\sqrt{\beta/xH)}_{max}$ and the solution is mainly concentrated in the narrow zone of the highest relative static stability, characterized by Brent-Wysel frequency. This zone falls approximately to altitudes 100-110 km. The layer of high stability is, thus, a sort of wave guide for high-frequency gravitational waves. (The second wave guide, where less high-frequency gravitational waves may concentrate, corresponds to the second peak of Brent-Wysel frequency at an altitude of 30 km). About this too, we shall speak later on.

7. Long waves.

The assumption that temperature rises limitlessly with altitude, has resulted in discreteness of the spectrum. However, in the same way as the analysis of equations for higher altitudes, this discreteness is of formal nature. Actually, solutions of equation (4.1) begin to vanish, when the value

$$\mu^2 = -\frac{1}{4} + \frac{\sigma^2 H}{xg}\left(1 - \frac{xH}{h}\right) + \frac{H\beta}{xgh} \qquad (4.24)$$

becomes negative (at sufficiently high altitudes this takes place due to terms $-\sigma H^2/gh$ on the strength of our assumption). However, for very low frequencies or for high values of equivalent depth h (i.e., for long waves) this occurs at a very high altitude, the analysis of which within the framework of our problem has no physical meaning. Assuming, for instance, that we are interested in the thickness of atmosphere upto 200 km., let us see, at what values of parameters, σ, h the vanishing begins not above altitude of 200 km, i.e., when the value of parameter μ^2 at altitude of 200 km is negative. In Fig. 4.5, below line 3 lies the region of parameters σ, h, for which $\mu^2(200) < 0$.

Let us analyse now in more detail, what happens in the region above line 3 in Fig. 4.5, in the zone of quasicontinuous spectrum. The energy now cannot be fully retained in the lower 200-kilometer layer, as it was for fully discrete spectrum. The energy penetrates to higher altitude. However, even here there are some thermal barriers, hindering the drift of energy to high altitudes. These are layers of the least μ^2 values. For long acoustical waves, i.e., at high h values, the last two terms in formula (4.24) could be disregarded and what we shall obtain is that the barriers are the layers with the least H values, i.e., the cold layers. Thus, we see here total inversion of position which had place for short waves. There the cold layers which were wave-guides, are here - barriers.

For long gravitational waves, at low σ, the terms may be disregarded in the equation containing σ, and we shall get as barrier layers with minimum values of product β H. Since β varies at considerably higher rate, than H, these layers approximately coincide with the minimum position of β at an altitude 60-70 km.

How effective are these thermal barriers? They cannot serve any more as an absolute obstacle for energy penetration into high layers, but can only retard this process. The discussion of this question could be conducted in the same way, as in quantum mechanics the study is made of the so called tunnel effect an event, very similar in form to the one under discussion. We should place in the lower part of the atmosphere a source of periodic oscillations, and at the top boundary z = 200 km to set a certain condition of the type of radiation, which should separate out of all resolutions those of the type of waves, propagating upwards, discarding waves propagating into opposite direction. Such problem for forced oscillations was analyzed by Wilkes for equations in quasistatic

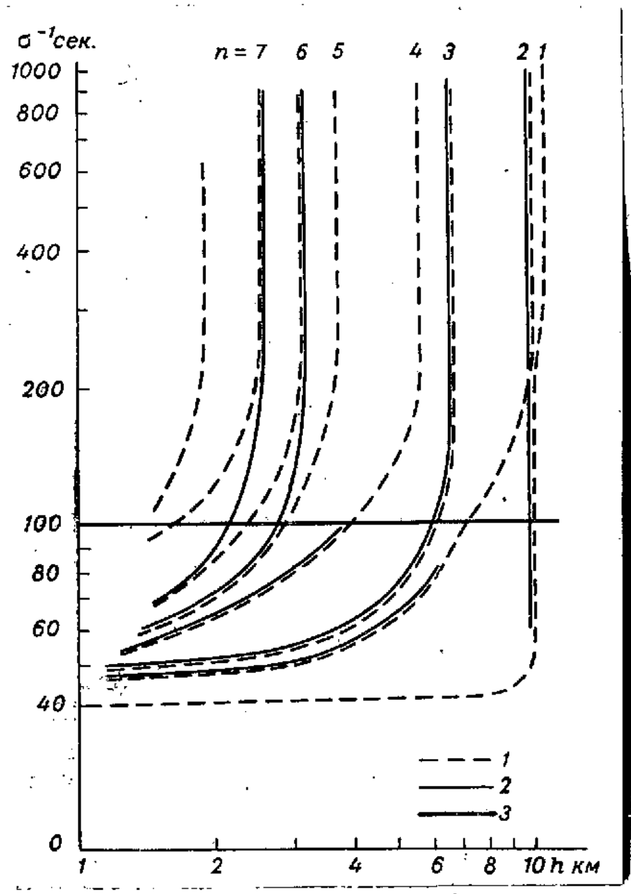

Fig. 4.5. Characteristic curves (1) and the curves of resonance peaks (2) of the problem on forced oscillations. 3 - limits of positive μ^2. region.

approximation and with a different temperature stratification. He took as the compelling forces the tide - generating forces, which are active mainly on the surface. It would be of interest to us to follow the way of the real spectrum conversion into spectrum of Wilkes type with increasing wave length.

Let us formulate conditions at the top limit. We assume, that $\mu^2(200) > 0$. Also, that above 200 km the factor μ^2 is constant. Then the equation (4.1) has two linearly independent solutions $e^{i\mu z}$ and $e^{-i\mu z}$. If it is taken into account that dependence on time in our problem has the form $e^{i\sigma t}$, it will be clear, that solution $e^{i(\sigma t- z)}$ is a wave propagating upward, and the solution $e^{i(t- z)}$ a wave propagating downward. Only the first one of these should be left. It is defined by the marginal condition

$$\frac{dy}{dz} = - i\mu y, \quad z = 200 \text{ km}. \tag{4.25}$$

In this paragraph we shall also analyse resolutions from the investigated zone $\mu^2(200) < 0$, but on condition

$$\frac{dy}{dz} = - \sqrt{- \mu^2 y}, \quad z = 200 \text{ km}. \tag{4.26}$$

This condition denotes exponential vanishing of solution above 200 km, if μ^2 is taken there as constant.

We introduce a fundamental system of equation (4.1) solutions. We assume

$$y(1) = 1, \quad y(2) = 0;$$

$$\frac{dy^{(1)}}{dz} = 0, \quad \frac{dy^{(2)}}{dz} = 1 \quad \text{at} \quad z = 200 \text{ km}.$$

Then the solution, which meets condition (4.25) at $\mu^2(200) > 0$, will be

$$y = y^{(1)} - i\mu \; - \; {}^{2}\, y^{(2)}.$$

Passing on to coercive force we account for it in the marginal condition

$$y'(0) - N(h)\,y(0) = \Omega \,, \quad N(h) = \frac{1}{2H(0)} - \frac{1}{h} \,, \qquad (4.27)$$

where Ω is the preset amplitude of disturbing force. If the term is considered for vertical velocity (1.38), the marginal condition (4.27) can be interpreted as the amplitude setting of vertical velocity on the earth's surface. In Wilke's work this condition for the case of semi-diurnal oscillations was obtained with an estimate of tide-generating potential. Denoting

$$M^{(1)}(h) = \frac{y^{(1)'}(0)}{y^{(1)}(0)} \,, \quad M^{(2)}(h) = \frac{y^{(2)'}(0)}{y^{(2)}(0)} \,.$$

Then at $\mu_*^2 > 0$

$$|\Omega|^2 = (M^{(1)}-N)^2 (y^{(1)}(0))^2 + \mu_*^2 \, (M^{(2)}-N)^2 (y^{(2)}(0))^2 ,$$

and at $\mu_*^2 < 0$

$$|\Omega|^2 = \left\{ (M^{(1)}-N) y^{(1)}(0) - \sqrt{-\mu_*^2} (M^{(2)}-N) y^{(2)}(0) \right\}^2 \,.$$

Now it is necessary to choose some measure, which would appraise the intensity of oscillation's excitation at a given amplitude of perturbing force. This measure, following Wilkes, we take the land pressure amplitude. From (1.19) and the term for vertical velocity (1.38) we find this amplitude (with accuracy up to constant factor, which is of no interest to us)

$$P^2 = \left| (\sigma^2 - \frac{g}{h})\, y + g\,(M - N) y \right|^2 \,.$$

Hence without difficulty we find pressure amplitude at $/\Omega/ = 1$:

a) at $\mu_*^2 > 0$

$$R = \frac{P|2|}{|\Omega|^2} = \frac{\left[\left(\frac{\sigma^2}{g} - \frac{1}{h}\right) + (M^{(1)}-N)\right]^2 \left[y^{(1)}(0)\right]^2 + \mu_*^2\left[\left(\frac{\sigma^2}{g} - \frac{1}{h}\right) + (M^{(2)}-N)\right]^2 \left[y^{(2)}(0)\right]^2}{(M^{(1)}-N)^2\left[y^{(1)}(0)\right]^2 + \mu_*^2 (M^{(2)}-N)^2\left[y^{(2)}(0)\right]^2}, \qquad (4.28)$$

b) at $\mu_*^2 < 0$

$$R = \frac{\left\{\left[\left(\frac{\sigma^2}{g} - \frac{1}{h}\right) + (M^{(1)}-N)\right] y^{(1)}(0) - \sqrt{-\mu_*^2}\left[\left(\frac{\sigma^2}{g} - \frac{1}{h}\right) + (M^{(2)}-N)\right] y^{(2)}(0)\right\}^2}{\left\{(M^{(1)}-N)y^{(1)}(0) - \sqrt{-\mu_*^2}\,(M^{(2)}-N)\, y^{(2)}(0)\right\}^2} \qquad (4.29)$$

The following estimate was carried out on computer: For series of wave numbers k, starting from sufficiently high (short waves) and ending with very low, R values were found and plotted on a curve (more correctly, on the curves given below was plotted the R/h value/sec Fig. 4.6) . Found on this curve were the resonance peaks. For short waves these resonance peaks coincide with the previously calculated eigen values h. With the increasing length of wave the resonance curves have peaks at frequencies, not necessarily coinciding with earlier estimated eigen values.

The general pattern is shown in Fig. 4.5. Here by line 2 are plotted the curves of resonance peaks, and by line 1 - characteristic curves at y'(200) = 0. Attention is attracted by the following features of this pattern. For modes n = 2, n = 3 the resonance peaks universally coincide exactly with eigen values. It should be mentioned that in this case the resonance peaks happen to be very energetic and acute. Curve 2, which initially coincides with mode n = 4, breaks off. Here, with increasing wave length the resonance is very acute, so that even insignificant h variation leads outside the limits of resonance peak. With this the height of peak decreases. After the passing of curve into region of positive μ_*^2 (boundary of this region is shown by curve 3) the resonance peak very quickly becomes indistinguishable.

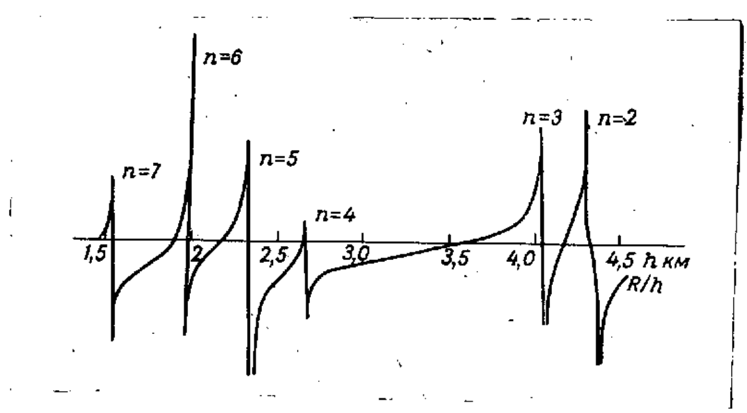

Fig. 4.6. Sample of resonance curve in the region of short gravitational waves. k = 0.082 km^{-1}.

The two following modes, n = 5 and n = 6, have a very interesting behavior. The resonance peaks here pass from one mode to the other adjacent one. The first of these, the one that passes from mode n = 5 to n = 6, is of high intensity and clearly defined, whereas the second is incompetent.

Fig. 4.6 - 4.8 show samples of resonance curves. Fig. 4.6 pertains to the region of discrete spectrum, Fig. 4.7 - to transitional zone, and Fig. 4.8 - to very long waves. This example is taken so that for mode n = 3 the period is half a day. This example was taken for comparison with Wilke's results. He studied conventionalized atmosphere, showing temperature curve by a broken line, given in Fig. 4.9. The resonance curve obtained by us is shown in Fig. 4.10. It is clear that the nature of curve is exactly the same, but quantitatively it is highly distinct from ours, obtained for standard atmosphere. There is a more important difference also - in position of mode n = 3. This is highly significant for the resonance theory of tides. The equivalent depth h = 6.7 km, obtained by us, is too far from that, which could confirm the resonance theory.

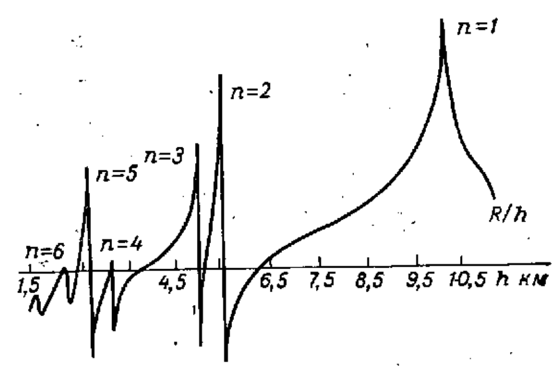

Fig. 4.7. Sample of resonance curve for transitional wave length. $k = 0.063\ km^{-1}$.

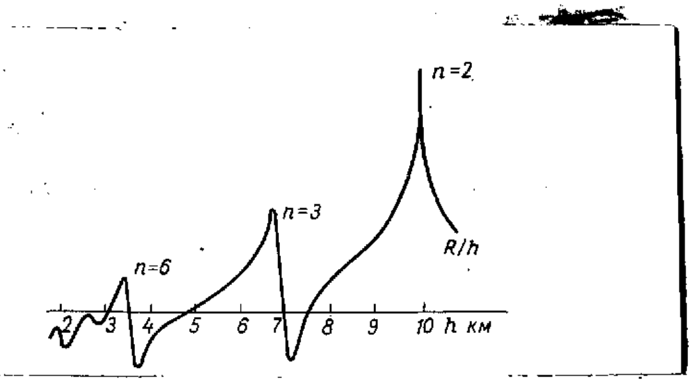

Fig. 4.8. Sample of resonance curve for long waves. $k = 0.57 \cdot 10^{-3}$.

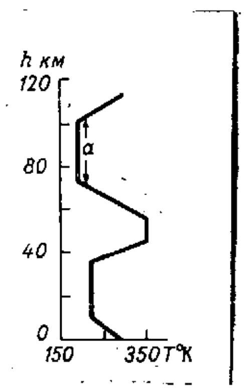

Fig. 4.9. Atmosphere model adopted in Wilke's book.

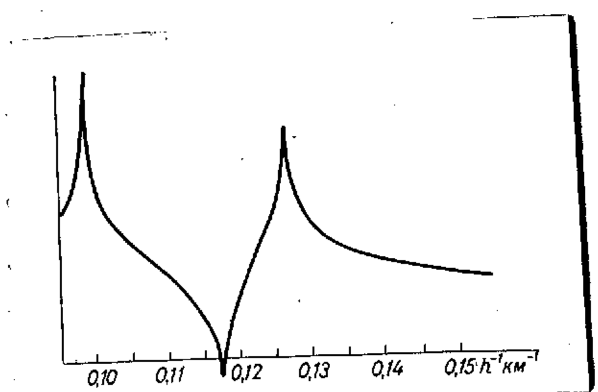

Fig. 4.10. Resonance curve, corresponding to model, shown in Fig. 4.9.

It should be mentioned, that from formula (4.28) one more conclusion may be arrived at. The high resonance peak is obtained in the case, when denominator is very close to zero. The latter means, that both the resolutions $y^{(1)}$ and $y^{(2)}$ at $x = 0$ meet highly accurately the boundary condition. In other words, eigen values of the problem with two different boundary conditions at the top limit are similar to each other. In this case the eigen values are very little dependent on the boundary condition at top limit.

Thus, the presence of cold barriers results in that at certain h values the atmosphere becomes hardly permeable for energy propagating upward, and a low in amplitude disturbing force can generate considerable oscillations. We are dealing only with periodical in time movements and not with the process of their setting. Apparently, an incompetent force can swing the atmosphere to a considerable extent only during a sufficiently long time.

8. Velocities of Rossbi waves. Comparison with empirical data.

The existence of Rossbi waves and formulas for their velocities were first theoretically shown by Rossbi and Haurwitz. It is extremely interesting to establish their existence empirically, from observations at Meteorological stations. The most perfect attempt of this kind was undertaken by Eliasen and Machenhauer (1965). By subjecting to harmonic analysis the field of pressure at different time moments, they found the velocity of phase shifting for individual harmonic components. These data were compared with theoretical velocities of Rossbi waves, obtained for incompressible atmosphere, i.e., at $\gamma = 0$. We know the following formula for frequencies of Rossbi waves in this case : $\sigma = 2\omega s/n(n + 1)$. Thus, phase velocity (or more correctly, angular phase velocity) $\varepsilon = - 2\omega/n(n + 1)$.

Eliasen and Machenhauer also took into account the mean angular velocity of rotation of the atmosphere's in relation to the earth α , i.e., the zonal transposition. Thus, the total angular velocity of rotation of the atmosphere's in absolute coordinates will be $\omega + \alpha$, this value should be inserted into formula for velocity instead of ω . Moreover, in order to obtain velocity not in relation to rotating atmosphere, but in relation to the system of coordinates bound up with the earth, addition should be made to velocity of α . We shall have, therefore,

$$\varepsilon = \alpha - \frac{2(\omega + \alpha)}{n(n + 1)} \quad . \tag{4.30}$$

In comparison of empirical results with those obtainable from formula (4.30), it is discovered that empirical velocities systematically obtained are lower than the theoretical. Eliasen and Machenhauer assumed, that the cause here is non-estimation of the compressibility of atmosphere's and suggested a method for the approximate estimate of compressibility. They used one empirical constant, with appropriate value of which satisfactory coincides with experiment.

It turns out, that it is possible to do without this approximate theory, if the values of natural frequencies are used for Laplace's tidal equation. Let us take for estimation $h = 10$ km, i.e., that which has place for the main complex mode. Then, with an estimate of α, taken by Eliasen and Machenhauer ($\alpha = 0.0225\,\omega$), corresponding value of parameter $\gamma = 4a^2 (\omega + \alpha)^2/gh$ will be 9.2. Table 4.1 gives velocity values of Rossbi waves ε_{obs}, obtained from observations, $\varepsilon_{incompr.}$, estimated from formula (4.30), and $\varepsilon_{compr.}$, determined from Laplace's tidal equation with an estimate of average zonal transposition. The table shows good concordance of ε_{obs} and $\varepsilon_{compr.}$. A more detailed discussion of this question could be found in Diky and Golitsyn article, (1968).

<u>T A B L E : 4 . 1</u>

Velocities of Rossbi waves (degree / day).

(n,s)	ε_{obs}	$\varepsilon_{incompr.}$	$\varepsilon_{compr.}$
(2,1)	- 70	- 115	- 64.0
(3,2)	- 40	- 53	- 40.3
(4,3)	- 20	- 28	- 24.7
(4,1)	- 20	- 28	- 21.5
(5,2)	- 12	- 16	- 13.4
(6,3)	- 8	- 9	- 8.0

CHAPTER : 5

ENERGY OF OSCILLATIONS.

1. Energy and classification of waves.

In the preceding chapter we saw, that all the oscillations could be classified in a certain way, by separating acoustical, gravitational and gygroscopic-inertia waves. The indication, according to which the classification was carried out, was to some extent formal : the behavior of these waves was investigated, with limit values of some parameters characterizing the structure and behavior of the atmosphere: parameter of static stability, parameter of compressibility and angular velocity of the earth's rotation.

These limit transitions can be carried out in a simple analytical model, in the case of isothermal atmosphere. For a model which is more complex and more approximating reality it would be impossible to change the parameters in this way, and not to lose the reality of the model. It is impossible, for instance, to direct the static stability parameter toward the zero. Thus, here the classification is based on simple analogy, on the fact, that the pattern of characteristic curves in a more complex case of real atmosphere is generally close in its nature to isothermal model.

However, it would have been considerably more interesting to clarify, whether there are structural and physical differences in waves of various types. The first objective characteristic of these differences could be the energy composition of a wave. We have seen in chapter 1, that the energy of atmosphere consists of several different parts. Due to anisotropy of the atmosphere it is expedient to analyse separately the kinetic energy, related to horizontal and vertical movements. As regards the potential energy here, too, it would be useful to separate two parts: energy related to compressibility, i.e., the elastic energy, and the energy related to stable lamination, i.e., according to terminology of Eckart, thermobaric energy.

It is natural to expect that acoustical and gravitational oscillations, i.e., quick waves, are related to periodic transition of energy from kinetic into potential and vice-versa. The kinetic energy comprises, on an average, 50% of the total energy (virial theorem). As regards the oscillations, under the effect of gyroscopic forces, which are not active, the kinetic energy here does not pass into potential and should be, therefore, absolutely predominant.

Hence, we should expect that acoustical and gravitational waves are distinct, one from the other, by the composition of potential energy. In the case of acoustical waves the predominant should be, apparently, pressure pulsations, i.e., elastic energy, and in the case of gravitational waves, pulsations of entropy, i.e., thermobaric energy. It should be mentioned that in spite of the great interest, evinced in regard to acoustical-gravitational waves, such an important question as the energetics of these waves, has been hardly touched by investigators. It is only possible to mention the work of Eliasen and Palm (1954), which dealt specially with energy of waves.

Let us investigate first the question regarding composition of energy of the waves theoretically. In para 4 we shall give results calculated on computer. A lot of attention will be paid not only to total characteristics of the energy, but also to energy distribution in altitude and in this connection to wave-guiding properties of the atmosphere.

In the space of vector-function $a = (u, v, w, p, \rho)$ it is possible to introdice a bilinear, positively determined form

$$\langle a_1, a_2 \rangle = \iiint \Big[\bar{\rho} \, \frac{u_1 u_2^* + v_1 v_2^* + w_1 w_2^*}{2} + \frac{1}{2x\bar{p}} \, p_1 p_2^* + \frac{g}{2x\bar{p}\beta} \, (p_1 - c^2 \rho_1)(p_2^* - c^2 \rho_2^*) \Big] \, dV. \quad (5.1)$$

Energy is a corresponding quadratic form. If there are two natural oscillations $e^{i\sigma_1 t} a_1$, $e^{i\sigma_2 t} a_2$ with different frequencies σ, the bilinear form $(e^{i\sigma_1 t} a_1, e^{i\sigma_2 t} a_2)$ is equivalent for those to zero, i.e., they are orthogonal in this matrix. In fact, the quadratic form for the sum of these oscillations $(e^{i\sigma_1 t} a_1 + e^{i\sigma_2 t} a_2, e^{i\sigma_1 t} a_1 + e^{i\sigma_2 t} a_2) = (a_1, a_1) + (a_2, a_2) + 2 \cos(\sigma_1 - \sigma_2)t \cdot (a_1, a_2)$ should not depend on time, since this is energy; therefore, $(a_1, a_2) = 0$.

But if two natural oscillations correspond to one and the same , but are different one from the other, they should have different values of h or γ. In this case our bilinear form for them is still zero on the strength of the orthogonality of the two solutions of Laplace's tidal equation at the same $f = \sigma/2\omega$, but different y [sec formula (2.5)]. Thus, the energy is a very convenient matrix, in which it is possible to study the fundamental solutions. In particular, it helps to expand the arbitrary solutions from fundamental, when the completeness of the latter will be proved in the next chapter. From the demonstrated orthognality it also follows, that the total energy of natural oscillations is equal to the sum of their energies, i.e., additive energy.

Now we shall write practically convenient formulas for the energy. Since all the variables are expressed through divergence, the energy can also be expressed through the same quantity. We recall formulas for the energy of four types (1.9 - 1.12). After y and ψ have been determined from equations, it is possible to find the values of unknown quantities, velocity components, pressure and density. It is easy to obtain these terms from equations in chapter 1. Thus, for pressure we have $p = p^*(x)\Psi$. Here $p^*(x)$ vertical component,

$$p^* = -\frac{-xg\bar{p}\left[H\sigma^2 y + g\left(y' - \frac{y}{2}\right)\right]}{\sqrt{\frac{\bar{p}}{\bar{p}_o}}\, i\sigma\left(\sigma^2 - \frac{g}{h}\right)},$$

Ψ - component dependent on horizontal coordinates. We shall analyse two cases, of spherical and of flat earth. In the first of these $\Psi = e^{is\varphi}\psi(\mu)$ where ψ - solution of Laplace's equation, and in the second even more simply $\Psi = e^{i(k_1 x_1 + k_2 x_2)}$. Similar terms we have for vertical velocity

$$w = \frac{- xg \left[y' + \left(\frac{H}{h} - \frac{1}{2} \right) y \right] \Psi}{\sqrt{\frac{\bar{p}}{\bar{p}o}} \left(\sigma^2 - \frac{g}{h} \right)} \tag{5.2}$$

and for entropy

$$p - c^2 \rho = - \frac{\beta \bar{\rho} w}{i \sigma}. \tag{5.3}$$

For horizontal velocity components we get the following terms. In the case of the spherical earth

$$u = \frac{- i \sigma p^* \xi \exp (is \varphi)}{4a\omega^2 \sqrt{1 - \mu^2} \bar{\rho}},$$

$$v = \frac{\mu p^* \left(\xi - \frac{s \Psi}{\mu f} \right) \exp (is \varphi)}{2a\omega \sqrt{1 - \mu^2} \bar{\rho}}, \tag{5.4}$$

where function ξ was determined in chapter 2 [sec system (2.3)] . For the flat earth we have $u = - k_1 p/\sigma \bar{\rho}$, $v = - k_2 p/\sigma \bar{\rho}$.

First, for simplicity, we shall analyse the case of the flat earth, which is admissible only in the study of sufficiently short waves. We find the energy of a vertical air column, section area of which is one. Substituting the terms just written into formulas for energy (1.9)--(1.12), we get the following relations [right portions are reduced $\varkappa \bar{p}_o / 2 \sigma^2 (\sigma^2 - g/h)^2$ times] :

$$E_r = \int_0^\infty \frac{xH}{h} \left[\sigma^2 y + g \left(y'_z - \frac{1}{2H} y \right) \right]^2 dz,$$

$$E_B = \int_0^\infty xgH\,\sigma^2 \left[y'_z + \left(\frac{1}{h} - \frac{1}{2H} \right) y \right]^2 dz,$$

$$E_y = \int_0^\infty \left[\sigma^2 y + g \left(y'_z - \frac{1}{2H} \, y \right) \right]^2 dz$$

$$E_T = \int_0^\infty g\beta \left[y'_z + \left(\frac{1}{h} - \frac{1}{2H} \right) y \right]^2 dz. \qquad (5.5)$$

The integrands densities of corresponding parts of energy, we denote by e_r , e_B , e_y , and e_T respectively.

2. Theorem of virial.

We demonstrate that on the assumption of flat non-rotatory earth kinetic energy (average during the period) is equal to potential, i.e., $E_r + E_B = E_Y + E_T$. For this the term for energy has to be converted. First of all we shall deal with kinetic energy. We integrate the first two formulas (5.5), by parts with consideration of equation (4.1) and boundary conditions (4.2)-(4.3). We shall not use the variable x, but the old variable z. Then the equation and the boundary conditions on the earth's surface will have the appearance:

$$y'' + \frac{H'}{H} y' + \left[- \frac{1}{4H^2} + \frac{2}{xg^H} \left(1 - \frac{xH}{h} \right) + \frac{\beta}{xghH} \right] y = 0, \qquad (5.6)$$

$$y' + \left(\frac{1}{h} - \frac{1}{2H} \right) y = 0 \text{ and } z = 0. \qquad (5.7)$$

Now, we convert the term for the horizontal component of kinetic energy, twice applying integration by parts and using (5.6 - 5.7):

$$E_r = \int_0^\infty \frac{xH}{h}\left[g^2(y')^2 + 2g\left(\sigma^2 - \frac{g}{2H}\right) y'y + \right.$$

$$\left. + \left(\sigma^2 - \frac{g}{2H}\right)^2 y^2 \right] dz = \int_0^\infty \frac{xH}{h}\left[-g^2 yy'' - \frac{g^2 H'}{H} yy' + \right.$$

$$\left. + 2g\left(\sigma^2 - \frac{g}{2H}\right) yy' + \left(\sigma^2 - \frac{g}{2H}\right)^2 y^2 \right] dz + \frac{xH}{h} g^2 yy' \Big|_0^\infty =$$

$$= \int_0^\infty \left\{ \frac{xH}{h}\left[-\frac{g^2}{4H^2} - \frac{\sigma^2 g}{xH}\left(1 - \frac{xH}{h}\right) + \frac{\beta g}{xhH} + \right.\right.$$

$$\left.\left. + \left(\sigma^2 - \frac{g}{2H}\right)^2 \right] y^2 - \frac{xg\sigma^2 H'}{h} y^2 \right\} dz +$$

$$+ \left[\frac{gxH}{h}\left(\sigma^2 - \frac{g}{2H}\right) y^2 + \frac{g^2 xH}{h} yy' \right]_0^\infty =$$

$$= \int_0^\infty \frac{xH}{h}\left[\sigma^4 - \frac{\sigma^2\beta}{xH} - \frac{\sigma^2 g}{h} - \frac{\beta g}{xhH} \right] y^2 \, dz -$$

$$- \left[\frac{xgH}{h}\left(\sigma^2 - \frac{g}{2H}\right) + \left(\frac{1}{2H} - \frac{1}{h}\right) \frac{xg^2 H}{h} \right] y^2(0) =$$

$$= \frac{x}{h}\left(\sigma^2 - \frac{g}{h}\right) \int_0^\infty H\left(\sigma^2 - \frac{\beta}{xH}\right) y^2 \, dz - \frac{xgH(0)}{h}\left(\sigma^2 - \frac{g}{h}\right) y^2(0).$$

Thus, we have

$$E_{\Gamma} = \frac{x}{h}\left(\sigma^2 - \frac{g}{h}\right)\int_0^{\infty} H\left(\sigma^2 - \frac{\beta}{xH}\right)y^2\,dz - \frac{xgH(0)}{h}\left(\sigma^2 - \frac{g}{h}\right)y^2(0). \tag{5.8}$$

Similarly, we convert the term for the vertical component of kinetic energy. We are giving only the final result, omitting all calculations

$$E_B = \sigma^2\left(\sigma^2 - \frac{g}{h}\right)\int_0^{\infty}\left(1 - \frac{xH}{h}\right)y^2\,dz. \tag{5.9}$$

For the elastic energy we get the following formula

$$E_y = g^2\int_0^{\infty}\left\{\frac{H'}{H}y'y + \left[\frac{\sigma^4}{g^2} + \sigma^2\left(\frac{1}{xgH} - \frac{1}{gh} - \frac{1}{gH}\right) - \frac{\beta}{xghH} - \frac{H'}{2H^2}\right]y^2\right\}dz + g\left(\frac{g}{h} - \sigma^2\right)y^2(0). \tag{5.10}$$

Here, in contrast to the two preceding cases, we did not integrate part by part the term containing yy'. The conversion of the term for thermobaric energy is somewhat more complex. Here, use will have to be made of one auxiliary relation. We multiply equation (5.6) by Hy' and then integrate, using the integration formula part by part we get

$$\int_0^{\infty} H'(y')^2 dz - \int_0^{\infty}\left[-\frac{1}{4H} + \frac{\sigma^2}{xg}\left(1 - \frac{xH}{h}\right) + \frac{\beta}{xgh}\right]' y^2\,dz -$$

$$-H(0)\left(\frac{1}{2H(0)}-\frac{1}{h}\right)^2 y^2(0) - \left[\frac{1}{4H(0)} - \frac{\sigma^2}{xg} \quad 1- \frac{xH(0)}{h} \quad + \right.$$

$$\left. + \frac{\beta(0)}{xgh}\right] \quad y^2(0) = 0. \tag{5.11}$$

Now we shall deal with the formula for thermobaric energy

$$E_T = \int_0^\infty \left[g\beta(y')^2 + 2g\beta\left(\frac{1}{h}-\frac{1}{2H}\right)y'y + g\beta\left(\frac{1}{h}-\frac{1}{2H}\right)^2 y^2\right] dz =$$

$$= \int_0^\infty \left[(x-1)g^2(y')^2 + xg^2H'(y')^2 - \left[g\beta\left(\frac{1}{h}-\frac{1}{2H}\right)\right]' y^2 + \right.$$

$$\left. + g\beta\left(\frac{1}{h}-\frac{1}{2H}\right)^2 y^2\right] dz - g\beta\left(\frac{1}{h}-\frac{1}{2H}\right) y^2(0).$$

Applying formula (5.11)

$$E_T = \int_0^\infty \left\{(x-1)g^2(y')^2 + xg^2\left[-\frac{1}{4H} + \frac{\sigma^2}{xg}\left(1-\frac{xH}{h}\right) + \right.\right.$$

$$\left.\left. + \frac{\beta}{xgh}\right]' y^2 - \left[g\beta\left(\frac{1}{h}-\frac{1}{2H}\right)\right]' y^2 + g\beta\left(\frac{1}{h}-\frac{1}{2H}\right)^2 y^2\right\} dz +$$

$$+ xg^2H(0)\left(\frac{1}{2H(0)}-\frac{1}{h}\right)^2 y^2(0) + xg^2\left[-\frac{1}{4H(0)} + \right.$$

$$\left. + \frac{\sigma^2}{xg}\left(1-\frac{xH(0)}{h}\right) + \frac{\beta(0)}{xgh}\right] y^2(0) - g\beta\left(\frac{1}{h}-\frac{1}{2H}\right) y^2(0).$$

Separately we convert the first term

$$\int_0^\infty (\varkappa - 1)g^2(y')^2 dz = -\int_0^\infty (\varkappa - 1)g^2 yy'' \, dz - (\varkappa - 1)g^2 y(0)y'(0) =$$

$$= (\varkappa - 1)g^2 \int_0^\infty \left\{ \frac{H'}{H} y'y + \left[-\frac{1}{4H^2} + \frac{\sigma^2}{xgH}\left(1 - \frac{xH}{h}\right) + \right.\right.$$

$$\left.\left. + \frac{\beta}{xghH} \right] y^2 \right\} dz - (\varkappa - 1)g^2 \left(\frac{1}{2H(0)} - \frac{1}{h} \right) y^2(0) =$$

$$= -\varkappa g^2 \int_0^\infty \left(\frac{H'}{2H}\right)' y^2 \, dz - g^2 \int_0^\infty \left\{ \frac{H'}{H} y'y - (x - 1)\left[-\frac{1}{4H^2} + \right.\right.$$

$$\left.\left. + \frac{\sigma^2}{xgH}\left(1 - \frac{xH}{h}\right) + \frac{\beta}{xghH} \right] y^2 \right\} dz - \varkappa g^2 \frac{H'}{2H} y^2(0) -$$

$$- (\varkappa - 1) g^2 \left(\frac{1}{2H(0)} - \frac{1}{h} \right) y^2(0).$$

Substituting these terms into formula for thermobaric energy and converting we finally obtain the following:

$$E_T = g^2 \int_0^\infty \left\{ -\frac{H'}{H} y'y + \left[\sigma^2 \left(\frac{1}{gH} - \frac{1}{xgH} - \frac{x}{gh} + \frac{1}{gh} - \frac{xH'}{gh} \right) - \right.\right.$$

$$\left.\left. - \frac{\beta}{xghH} + \frac{H'}{2H^2} + \frac{\beta}{gh^2} \right] y^2 \right\} dz +$$

$$+ g^2 \left(\frac{\sigma^2}{g} - \frac{1}{h} \right)\left(1 - \frac{xH(0)}{h}\right) y^2(0). \qquad (5.12)$$

Now without any difficulty we find

$$E_y + E_T = \int_0^\infty \left[\sigma^4 - \sigma^2 \frac{\beta + g}{h} + \frac{\beta g}{h^2}\right] y^2 \, dz -$$

$$- \left(\sigma^2 - \frac{g}{h}\right) \frac{xH(0)g}{h} y^2(0).$$

Adding up the terms for horizontal and vertical components of kinetic energy we get the same

$$E_r + E_B = \int_0^\infty \left(\sigma^2 - \frac{g}{h}\right)\left(\sigma^2 - \frac{\quad}{h}\right) y^2 \, dz - \left(\sigma^2 - \frac{g}{h}\right) \frac{xH(0)g}{h} y^2(0).$$

Thus, the theorem regarding parity of kinetic and potential energy is fully proved.

In proving this theorem it was essential to assume, that the earth is flat and non-rotatory. Otherwise the theorem could simply not be true. If the gyroscopic forces are of significance, the share of kinetic energy should increase; if we take only the gyroscopic-inertia waves, the kinetic energy does not pass into potential energy at all since the forces in this case are inactive.

Let us try and estimate how much the share of kinetic energy increases due to the earth's rotation. This requires writing new formulas for energy on the spherical rotatory earth. In this case the field is not uniform in horizontal coordinates, and we cannot confine ourselves to calculating energy of vertical air column of singular section. We must find total energy throughout the atmosphere. By using (5.4) we write a formula for horizontal energy

$$E_r = \frac{2\pi}{2}\int_{-1}^{1}\int_{0}^{\infty}\frac{\sigma^2\xi^2+4\omega^2\mu^2(\xi-\frac{s\psi}{\mu f})^2}{16a^2\omega^4(1-\mu^2)\bar{\rho}}(p^*)^2 dz\, d\mu =$$

$$= \frac{\varkappa^2 g\pi\bar{p}_o}{16a^2\omega^4(\sigma^2-\frac{g}{h})^2}\int_0^{\infty} H\left[\sigma^2 y+g\left(y'-\frac{1}{2H}y\right)\right]^2 dz \times$$

$$\times\int_{-1}^{1}\left[\xi^2+\frac{\mu^2}{f^2}\left(\xi-\frac{s\psi}{\mu f}\right)^2\frac{d\mu}{1-\mu^2}\right.$$

The other energy components we get in the same way:

$$E_B = \frac{\pi\varkappa^2\bar{p}_o g}{(\sigma^2-\frac{g}{h})^2}\int_0^{\infty} H\left[y'+\left(\frac{1}{h}-\frac{1}{2H}\right)y\right]^2 dz\int_{-1}^{1}\psi^2 d\mu,$$

$$E_y = \frac{\pi\varkappa p_o}{(\sigma^2-\frac{g}{h})^2\sigma^2}\int_0^{\infty}\left[\sigma^2 y+g\left(y'-\frac{y}{2H}\right)\right]^2 dz\int_{-1}^{1}\psi^2 d\mu,$$

$$E_T = \frac{\pi\varkappa g\bar{p}_o}{(\sigma^2-\frac{g}{h})^2\sigma^2}\int_0^{\infty}\beta\left[y'+\left(\frac{1}{h}-\frac{1}{2H}\right)y\right]^2 dz\int_{-1}^{1}\psi^2 d\mu.$$

Hence we discern without difficulty, that the vertical, elastic and thermobaric energies are proportional to those values, which they had in the case of the flat non-rotatory earth. As regards the horizontal energy, it is multiplied by additional multiple

$$= \frac{\gamma^{-1} \int\limits_{-1}^{1} \left[f^2 \xi^2 + \mu^2 \left(\xi - \frac{s\psi}{\mu f} \right)^2 \right] \frac{d\mu}{1 - \mu^2}}{\int\limits_{-1}^{1} \psi^2 \, d\mu}$$

Let us find out by how much this multiple differes from a unit. For this we must recall formula (2.4), whence

$$\rho - 1 = \frac{2\gamma^{-1} \int\limits_{-1}^{1} \left(\mu^2 \xi^2 - \frac{s\mu}{f} \xi \psi \right) \frac{d\mu}{1 - \mu^2}}{\int\limits_{-1}^{1} \psi^2 \, d\mu} . \tag{5.13}$$

This multiple is appraised most simply, when it is possible to use asymptotes of Laplace's tidal equation for high γ^{-1}. For oscillations of the first kind (in the present case for gravitational waves) we have

$$f^2 \xi \approx (1 - \mu^2) \psi' .$$

Therefore,

$$\rho - 1 = \frac{2\gamma^{-1} \int\limits_{-1}^{1} \left(\frac{\mu^2}{1 - \mu^2} \xi^2 + \frac{s}{2f^3} \psi^2 \right) d\mu}{\int\limits_{-1}^{1} \psi^2 \, d\mu} . \tag{5.14}$$

For oscillations of the second kind (gyroscopic-inertia waves)

$$\frac{s^2 \psi}{f^2} \qquad (1 - \mu^2) \xi' + \frac{s \mu \xi}{f} .$$

Therefore,

$$\rho - 1 = \frac{\gamma^{-1} \frac{f}{s} \int\limits_{-1}^{1} \xi^2 \, d\mu}{\int\limits_{-1}^{1} \psi^2 \, d\mu} . \tag{5.15}$$

It is evident, that in both the cases $\rho > 1$, i.e., actually due to rotation there is an increased share of horizontal kinetic energy. The last two formulas could be even more firmly based, if an asymptotic value is substituted in the first one

$$\psi \sim P_n^s \, , \; \xi \sim \frac{1}{f^2} \left[- \frac{n(n - s + 1)}{2n + 1} P_{n+1}^s + \right.$$

$$\left. + \frac{(n + 1)(n + s)}{2n + 1} P_{n-1}^s \right] , \tag{5.16}$$

and in the second

$$\frac{s}{f} = n(n + 1), \; \xi \sim P_n^s, \; \psi \sim \frac{n - s + 1}{(2n + 1)(n + 1)^2} P_{n+1}^s +$$

$$+ \frac{n + s}{(2n + 1) \, n^2} P_{n-1}^s . \tag{5.17}$$

Calculations from formula (5.14) are rather difficult. Instead of this it would be better to present the main formula (5.13) in another way and then to substitute there the asymptotic values of fundamental functions. Equation

$$(1-\mu^2)\,\xi' + \frac{s\mu}{f}\,\xi = \left[\frac{s^2}{f^2} - (1-\mu^2)\,\gamma\right]\psi$$

second of the equations in system (2.3) we multiply by $\mu\,\xi/(1-\mu^2)$ and integrate. We will have

$$\frac{s}{f}\int_{-1}^{1}\left(\mu^2\xi^2 - \frac{s\mu}{f}\xi\psi\right)\frac{d\mu}{1-\mu^2} = \int_{-1}^{1}\left(\frac{1}{2}\xi^2 - \gamma\mu\xi\psi\right)d\mu.$$

Hence, and from (5.13) we get

$$\rho - 1 = \frac{2\gamma^{-1}\frac{f}{s}\int_{-1}^{1}\left(\frac{1}{2}\xi^2 - \gamma\mu\xi\psi\right)d\mu}{\int_{-1}^{1}\psi^2\,d\mu}.$$

Substituting here asymptotes (5.16) we find

$$\rho - 1 = \frac{1}{fs(2n+1)}\left[\frac{(n-s+1)(n+s+1)n}{2n+3}\left(\frac{n\gamma^{-1}}{f^2} + 2\right) + \right.$$

$$\left. + \frac{(n-s)(n+s)(n+1)}{2n-1}\left(\frac{(n+1)\gamma^{-1}}{f^2} - 2\right)\right]. \qquad (5.18)$$

For the waves of the second kind we substitute asymptotes (5.17) in formula (5.15). We shall have

$$\rho - 1 = \frac{\gamma^{-1}}{\dfrac{n(n-s+1)(n+s+1)}{(n+1)^3(2n+1)(2n+3)} + \dfrac{(n+1)(n+s)(n-s)}{n^3(2n-1)(2n+1)}}. \qquad (5.19)$$

Let us take the long gravitational waves, corresponding to atmospheric tides. Here n = s = 2, f = 1. As shown in chapter 2, to semi-diurnal oscillations corresponds to $\gamma^{-1} \approx 0.09$. Substituting all this in formula (5.18), we find that $\rho - 1 = 0.3$, $\rho = \ = 1.3$. Thus, in this case the amplification factor of horizontal kinetic energy due to the earth's rotation is negligible. We shall see further, that for such long gravitational waves the share of vertical energy is negligibly low, i.e., the whole kinetic energy consists of horizontal motion energy. Hence it follows that the kinetic energy of tidal waves composes 57% of the whole energy, and the potential 43%.

An entirely different pattern is obtained for the waves of the second kind, for instance, for the two-dimensional Rossbi waves. Let us take, for example, the same values $n = s = 2$, $\gamma^{-1} = 0.1$, and $\rho - 1 = \ = 10$. The amplification factor is found to be so high, that practically the whole energy ($> 90\%$)could be taken as kinetic. The share of potential (thermobaric) energy increases only at very high γ (low h), i.e., in waves of large horizontal and low vertical scales.

The theorem of virial is known in the general mechanics of the material points system. But this does not relieve us of the necessity to prove it in our concrete case, since, firstly, it is not always fulfilled, but as shown by the reasons given above, only in certain conditions, which need checking. Secondly, although, for example, the law of energy preservation is a universal law of nature, nevertheless, in mechanics of continuous media it is checked again, by way of not quite trivial calculations (which, perhaps, is the checking of the common-sense of the main equations). An almost independent fact is the existence of quadratic invariant - energy for linearized equations, which happen to be approximate equations. In any case this fact requires proof. Exactly the same applies to the theorem of virial.

3. Energy composition of oscillations.

We turn again to formulas (5.5) for energy of various types. Formulas for densities of horizontal and elastic energy differ only by the multiple $\varkappa$ H/h. Hence, it is clear that with h increase the energy composition of oscillations should change toward the increasing share of elastic energy, as against the horizontal. If we take a look of the main chart of characteristic curves (see Fig. 4.4), we shall see, that the share of elastic energy increases to the side of acoustical waves and decreases toward the gravitational waves, which, of course, is quite consistent with the physical meaning.

Exactly in the same way, comparing vertical and thermobaric energy, we not that their densities differ by multiple σ^2/σ_v^2, where $\sigma_v^2 = \beta/\varkappa H$ - Brent - Wysel frequency for the given altitude. Thus, the vertical energy increases in comparison with thermobaric, when the frequency increases in comparison with Brent-Wysel frequency. Vertical energy is comparable with thermobaric in the range of frequencies similar to mean Brent-Wysel frequencies. From the Fig. it is obvious, that the share of thermobaric energy increases toward the gravitational waves, and that of vertical energy - toward acoustical waves, which is again concurrent with instinctive physical reasoning.

It is possible to write a rough estimate

$$E_{\Gamma} = \frac{\varkappa \bar{H}}{h} E_y \ , \quad E_B = \frac{\sigma^2}{\bar{\sigma}_v^2} E_T \ , \tag{5.20}$$

where, H, σ_v^2 are certain average values of corresponding quantities. Now, we add up these parities taking into account that $E \ + E_B = E_y + E_T$:

$$\left(\frac{h}{\varkappa H} - 1 \right) E_{\Gamma} + \left(\frac{\bar{\sigma}_v^2}{\sigma^2} - 1 \right) E_B = 0. \tag{5.21}$$

This relation fixes the bond between E_{Γ} and E_B.

In the region of h, similar to average $\varkappa$ H values, the predominant should be the horizontal energy, and in the region of σ^2, similar to σ_v^2 - vertical. However, this relation is very rough, since H and σ_v^2 vary in rather a wide range, and the H and σ_v^2 quantities remain not very definite.

It is remarkable, however, that it is possible to indicate an absolutely accurate and simple relation between the vertical and horizontal energy, which permits the estimation of the share of these two types of energy directly on the chart of characteristic curves. For deduction of this relation we shall use other terms for the energy components, deduced in preceding para (5.8),(5.9). Now let us reason in this way. Let us find the slope of the tangent to characteristic curve in Fig. 4.4. Assuming that we shifted along this curve from point (h, σ) to point (h + dh, σ + dσ), the solution y depends on quantities σ and h as on parameters and with each value of these parameters along the curve meets the equations and marginal conditions. We denote dy = η . Then, by differentiating along the curve equation (4.1), we shall have

$$\eta'' + \left[- \frac{1}{4} + \frac{\sigma^2 H}{\varkappa g} \left(1 - \frac{\varkappa H}{h} \right) + \frac{H\beta}{\varkappa g h} \right] \eta =$$

$$= - \frac{H}{\varkappa g} \left(1 - \frac{\varkappa H}{h}\right) y \, d\sigma^2 + \frac{H}{\varkappa g h^2} \left(\beta - \varkappa \sigma^2 H\right) y \, dh. \qquad (5.22)$$

Similarly, differentiating the marginal condition (4.2), we shall get

$$\eta' + \left(\frac{H}{h} - \frac{1}{2} \right) \eta = \frac{Hy \, dh}{h^2} \qquad (5.23)$$

(thedash above the variable means, as in chapter 4, that differentiation was not by z, but by x). We multiply (4.1) by η , (5.22) by y, substract and integrate from 0 to ∞ . Considering, that dz = Hdx, we shall have, after partial integration, and taking into consideration the boundary conditions (4.2) and (5.23)

$$\frac{H(0)}{h^2} y^2(0) dh = \int_0^\infty \frac{1}{\varkappa g} \left(1 - \frac{\varkappa H}{h} \right) y^2 \, dz \, d\sigma^2 -$$

$$- \int_0^\infty \frac{1}{\varkappa g h^2} \left(\beta - \varkappa \sigma^2 H \right) y^2 \, dz \, dh.$$

If the result obtained is compared with formulas for energy (5.8),(5.9), it would be possible to obtain an exceptionally simple relation between the energies

$$E_r \frac{dh}{h} + E_B \frac{d\sigma^2}{\sigma^2} = 0 \qquad (5.24)$$

(with the use of logarithmic variables $h_1 = \ln h$, $\sigma_1 = \ln \sigma^{-2}$ it would look specially simple : $d\sigma_1 / dh_1 = E_{\text{г}} / E_{\text{в}}$). The ratio between horizontal and vertical energy is determined by the slope of characteristic curve.

Hence, incidentally, we shall obtain, absolutely unexpectedly, one more general conclusion regarding the position of characteristic curves of equation (4.1)- their monotony. Increase of h causes increase of σ^{-1}.

The relation (5.24) admits even physical interpretation. For this a concept should be brought in of horizontal gourp velocity, or the propagation velocity of energy horizontally. Horizontal group velocity $c_{\text{гр}} = d\sigma / dk$. Now let us recall that qualities σ , h and k are bound by relation $\sigma / k = \sqrt{gh}$. We differentiate this relation along the characteristic curve

$$d\sigma = dk \sqrt{gh} + \frac{1}{2} k \sigma \frac{dh}{h} ;$$

hence, even from (5.24), by excluding dh we get

$$\frac{d\sigma}{dk} = \frac{E_{\text{г}}}{E_{\text{г}} + E_{\text{в}}} \frac{\sigma}{k} ,$$

or

$$c_{\text{гр}} = \frac{E_{\text{г}}}{E_{\text{г}} + E_{\text{в}}} c_{\phi} , \qquad (5.25)$$

where c_{ϕ}—is phase velocity of wave propagation horizontally. Thus, the group velocity is always less than phase velocity by as many times as horizontal energy is less than total kinetic energy.

By analyzing the charts of characteristic curves (see Fig. 4.4), we can say now, that in those areas, where the curve has a vertical direction or very close to it, the main portion of kinetic energy is the horizontal. But where the curve approaches horizontal direction, the predominant is the vertical energy. In the first case the group velocity is almost a phase velocity, in the second it is much lower than the phase velocity. Thus, for the gravitational bunch in its lower portion - for short (high - frequency) gravitational waves the share of vertical energy is high. Gravitational waves correspond to low h, i.e., low phase velocities $\sqrt{gh}$. Moreover, the group velocity of short gravitational waves is much lower than the phase velocity. Hence it follows, that the group velocities of these waves are very low. Their horizontal propagation is very slow, and the motion of particles in them occurs predominantly vertically. With increasing wave length group velocity becomes higher.

The acoustical waves could have as high phase velocities as desired. But the higher the phase velocity, the more horizontal is the curve, i.e., the ratio of group velocity to phase velocity decreases. As a result the highest group velocity is not where the phase velocity is highest, but where the characteristic curve has vertical direction, i.e., for curve $h \approx 10$ km, or for the main complex mode. Here, the group velocity coincides with phase velocity and is 315 m/sec. This velocity is the same for all frequencies, starting from some maximum. There is practically no dispersion of waves here.

4. Results of numerical calculations. Atmospheric wave guides.

The reasoning of the preceding paragraphs is confirmed by numerical calculations of energy on computers. Simultaneously with calculations of characteristic curves values were obtained of various types of energy. Table 5.1 gives these values for some natural oscillations, selected, for example, from various sections of the spectrum. Fig. 5.1 shows reproduced from Fig. 4.4 characteristic curves, and marked on them are the points, corresponding to examples given in the table. The contents of Table 5.1 are as follows. Column 1 gives the number of example, column 2- number of the wave's mode, i.e., the number of modes in function y. This number depends, of course, on the boundary condition taken at the top limit z = 200 m. If a different condition had been taken, for instance y = 0, or demanded conversion into zero of vertical velocity, the number of modes could have changed by one.

TABLE : 5.1

Energy composition of waves.

Number of example.	Number of mode	h km	σ^{-1} sec.	E_r, %	E_B %	E_y, %	E_Γ %	c_Φ m/sec	c_{rp} m/sec
1	2	3	4	5	6	7	8	9	10
1	9	1.6	13790	50	0	6	44	125	125
2	8	2.0	12530	50	0	8	42	140	140
3	7	2.6	10920	50	0	11	39	160	160
4	6	3.2	9690	50	0	15	35	178	178
5	5	3.9	8860	50	0	15	35	197	197
6	4	5.9	7212	50	0	21	29	242	242

Number of example.	Number of mode	h km	σ^{-1} sec.	E_r %	E_θ %	E_y, %	$E_{r'}$ %	c_Φ m/sec	c_{rp} m/sec
1	2	3	4	5	6	7	8	9	10
7	3	6.8	6780	50	0	32	17	260	260
8	2	10.0	5550	50	0	50	0	314	314
9	6	3.1	188	46	4	12	38	175	161
10	5	3.5	176	44	6	12	38	186	163
11	4	5.1	145	40	9	16	35	225	180
12	3	6.3	131	47	4	28	21	247	232
13	5	2.3	81	32	18	7	43	151	97
14	4	2.7	75	29	21	6	44	164	95
15	3	4.1	61	27	23	10	40	202	108
16	2	4.4	59	25	25	10	40	210	105
17	3	0.56							
18	2	3.2							
19	0	8.63	12.8						
20	1	9.5	9.1						
21	2	9.6	9.1						
22	3	10.5	8.7						
23	4	80.9	8.6						
24	1	9.52	9.13	50	0	50	0	307	307
25	2	9.46	8.35	50	0	50	0	306	306
26	1	9.49	8.71	46	4	50	0	307	282

Number of example.	Number of mode	h km	$^{-1}$ sec.	E , %	E , %	E_y, %	E , %	c m/sec	c m/sec
1	2	3	4	5	6	7	8	9	10
27	1	9.60	9.10						
28	0	8.95	40	50	0	50	0	314	514
29	1	21.2	38	15	35	31	19	456	135

Columns 3 and 4 show values of natural parameters h and σ^{-1}. Period T is equal to $2\pi\sigma^{-1}$. Thus, the period, given in minutes, composes approximately 1/10 σ^{-1}, given in seconds. Columns 5 - 8 give energy values of oscillations in a vertical column of atmosphere. They are given not in absolute values, but in percentage to total energy. It should be emphasised, that these values are calculated from formulas (5.5), i.e., for a flat non-rotatory model. As we already know in the estimate of rotation the share of kinetic horizontal energy highly increases, specially for the long-period waves of the second kind.

Finally, columns 9 - 10 indicate values of phase and group velocities. In principle these concepts are applicable also to not very long waves and were obtained from the formulas known to us

$$c_\Phi = \sqrt{gh}, \quad c_{rp} = c_\Phi \frac{E_r}{E_r + E_B} .$$

The examples in the table are grouped in series: series of very long period, about half a day, gravitational waves, or series of short-period waves, series of acoustical waves, etc.

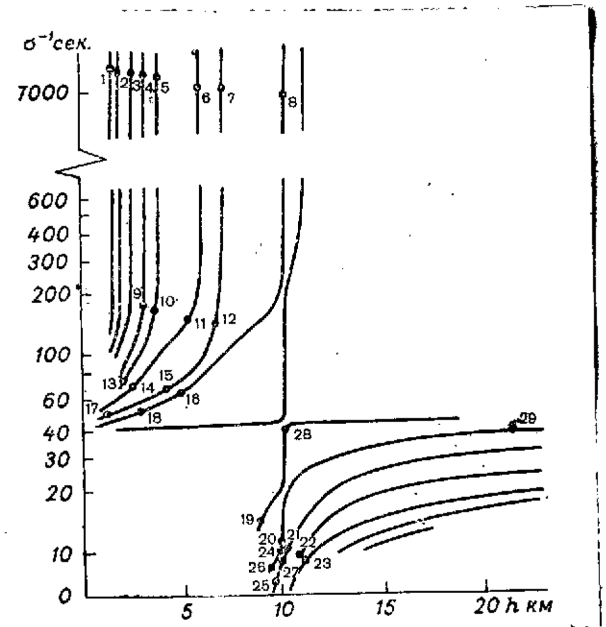

Fig. 5.1. Points, corresponding to examples, shown in Tab. 5.1.

As we have mentioned, the results of calculations illustrate the facts proved earlier. The fact, that in all case kinetic energy is equal to potential, is simply the property of those formulas, from which the calculation was being carried out, and can serve now as confirmation of the accuracy of calculations.

For those sections of characteristic curves, which have vertical direction, the energy of vertical motion is insignificantly low in comparison with energy of horizontal motion. This applies to all waves with $h \approx 10$ km, and also to all long-period gravitational waves. The share of the thermobaric energy of the latter, in spite of the lowness of vertical velocities, is high. This is caused by extremely low frequency in cimparison with characteristic magnitude of the Brent-Wysel velocity (two orders in our examples).

However, for one of these modes, corresponding to h = 10 km, even the thermobaric energy is zero. This indicates, that here the vertical velocity is immeasurably less than in other cases (this circumstance is not shown in the table, as the energy values are given with accuracy up to whole numbers). The entire series h = 10 km is distinct by high degree of two-dimensionality, total absence of vertical velocity, as in the case of isothermal atmosphere.

The share of thermobaric energy of short gravitational waves (examples 9-18) is very high and the vertical velocity here reaches high values, specially in the short-period waves. In examples 17,18, relating to gravitational waves with frequencies close to limiting, vertical energy considerably exceeds horizontal. In short gravitational waves the movement of air particles is mainly vertical. Group velocity of these waves, i.e., the rate of horizontal transportation of energy, is, naturally, very low.

In accordance with the general position, the elastic energy increases with the rise of phase velocity, i.e., with h increase. And in this case it is found, that the elastic energy of low mode gravitational waves (examples 6,7,8) is not low at all; this speaks of certain conventionality: in this case of the name " gravitational waves ". The share of elastic energy of acoustical waves is always considerable. The share of thermobaric energy could also be appreciable, if the frequency is not too high, and phase velocity considerable, which is related to the presence of noticeable vertical velocities (example 29).

Finally, once more we pay attention to the fact that the highest group velocity has the two-dimensional waves of series h = 10 km.

A lot of interesting facts could be learnt regarding properties of natural oscillations by studying energy distribution of oscillations in altitude. Those of long-period oscillations of the first series (examples 1 - 8) have physical meaning, energy of which is concentrated in the lower layers of the atmosphere, or at least vanishes in sufficient measure before the limit of 200 km. The energy of these oscillations is retained by the temperature lamination of the atmosphere. This separates those modes, which coincide with resonance amplification in the problem of forced oscillations of atmosphere. Fig. 5.2 - 5.5 show distribution curves of energy with altitude for examples 4, 5, 7, 8. We see, that for the first two of these, pertaining to modes 6 and 5, the energy vanishes with altitude, but not very quickly. There is an interesting regularity - the thermobaric energy attains highest values, where the kinetic energy is low (minimum) and vice-versa. In any case the theorem of virial is not fulfilled at every point of space, but only integrally, along a column.

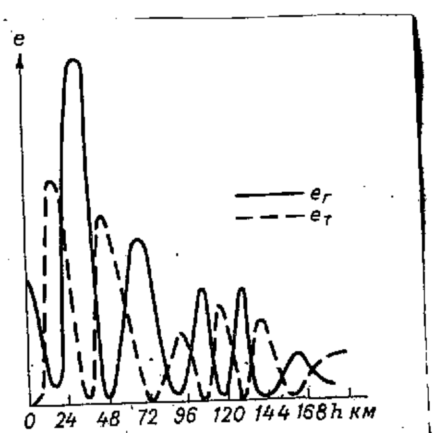

Fig. 5.2. Distribution of energy in height. Example 4.

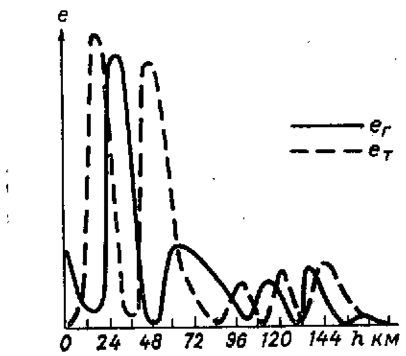

Fig. 5.3. Distribution of energy in height. Example 5.

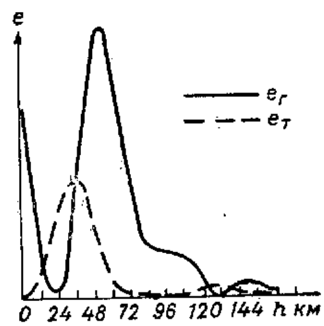

Fig. 5.4. Distribution of energy in height. Example 7.

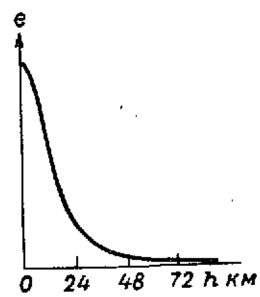

Fig. 5.5. Distribution of energy in height. Example 8.

Considerably a sharp decay is seen on the next two figures for modes 3 and 2. This is not incidental. We saw even before, that these modes give very acute resonance amplification. Mode 3 is of great interest in the theory of tides, as it pertains to h value, closest to that, which corresponds to semi-diurnal tides, (as shown in chapter 3 this h = 7.9 km). The attention is drawn on the curve to the peak of kinetic energy at an altitude of about 55 km and the less considerable peak on the earth's surface. The elastic energy is distributed in the same way whereas the thermobaric has its peak at an altitude of 30 km. Above 100 km the share of energy is insignificant. Thus, it may be assumed, that the energy of tidal oscillations is retained mainly in the lower 100 km.

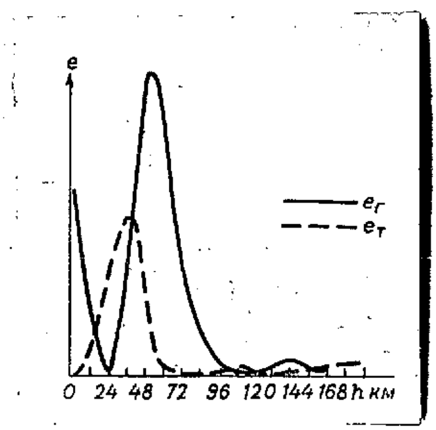

Fig. 5.6. Distribution of energy in height. Example 12.

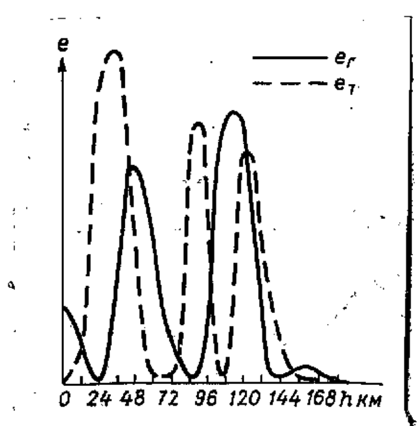

Fig. 5.7. Distribution of energy in height. Example 15.

An entirely different nature has oscillation, corresponding to mode 2 (Fig. 5.5). Here, the oscillations areconcentrated in surface region. The energy very quickly drpps with height. This once again emphasizes similarity of oscillations at h = 10 km with the two-dimensional oscillations of isothermal atmosphere. This behavior is inherent in all the oscillations at h = 10 km regardless of their periods, starting with period of about 10 min and higher. As regards the higher-frequency oscillations, we shall speak of them later.

Figs. 5.6 - 5.7 pertain to examples 12 and 15, i.e., to the same mode n = 3, as the tidal oscillations, but not of very high periods. Example 12 corresponds to period of about 13 min. But the nature of oscillations here remains the same, as the nature of semi-diurnal tidal oscillations. Here the quasistatic approximation still depicts the oscillations qualitatively rather well, although there is some quantitative shift (h = 6.3, and not 6.8). For smaller periods, for instance 6 min in example 15 (Fig. 5.7), the pattern is considerably different from this. Here the thermobaric energy and even the vertical are highly significant. On the whole the energy is more diffused in height.

Fig. 5.8 and 5.9 pertain to examples 14, 17 and 18. All of them are characterized by low h values. We know, that short gravitational waves with low h values have the tendency to concentrate in wave guiding layers. There are two of these wave guides in the atmosphere: deep wave guide in the zone of maximum static stability at height of 110 km and a less deep wave guide at a height of 30 km. In example 17 the oscillations are concentrated in the lower wave-guide, in examples 14 and 18 - in the upper. Examples 17 and 18 pertain to shorter waves, and the concentration within the wave guides is tighter. Moreover, example 14 corresponds to higher mode (4); therefore, the energy curve has a more dissected view (high number of peaks).

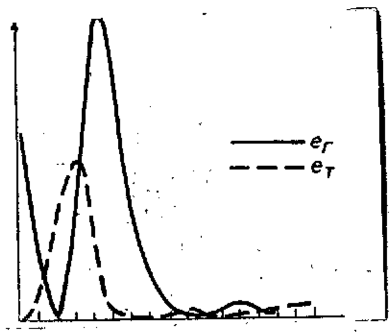

Fig. 5.8. Distribution of energy in height. Examples 14(a) and 17(b).

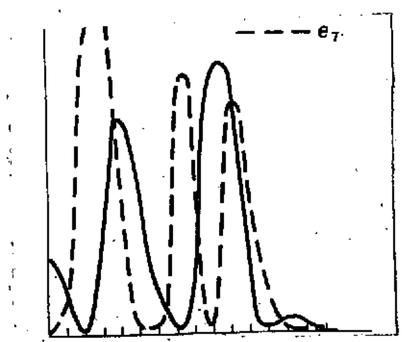

Fig. 5.9. Distribution of energy in height. Example 18.

Let us pass on to acoustical waves. As shown by Fig. 5.1, our examples pertain to a small area of spectrum - to oscillations with period of about 1 min of the first modes. This area is rather interesting. Here begins the appearance of short - wave asymptotes. For the short acoustical waves there are two wave guides in the atmosphere : deep wave guide in the cold layer in the mesosphere at an altitude of about 84 km and a less deep wave guide in the stratosphere at a height of about 17 km. The shortest waves should concentrate in the deep mesosphere wave guide, less short could be even in the stratosphere.

As mentioned in the preceding chapter, there is a composite mode $h \approx 10$ km, consisting of separate portions of characteristic curves. This was first discovered by Press and Harkrider (1962) and Pfeffer and Zarichny (1963). It is possible that each characteristic curve with decrease of period becomes for a time a part of this complex curve, so that with further decrease of period to withdraw from it to the left and toward the limit $h = \varkappa H_{min}$. At least for those characteristic curves, for which we carried out calculations, the following regularity defines itself. For portions of characteristic curves, which make up the composite curve, the corresponding natural oscillations are concentrated in the lower wave guide. When the characteristic curve with decrease of period withdraws to the left, the energy passes into upper wave guide and remains there with further decrease of period. Let us take, for instance, Fig. 5.10a and 5.10b (examples 24 and 25). They appear absolutely identical, whereas the first one pertains to mode 1, and the second to mode 2. But the first one corresponds to slightly higher period, when mode 1 was still the part of the complex mode $h \approx 10$ km, but in the second case, for slightly smaller period, the modes have shifted: mode 1 moved away to the left, and was replaced by mode 2. The fundamental function ψ now has one more mode, whereas the energy curve looks the same.

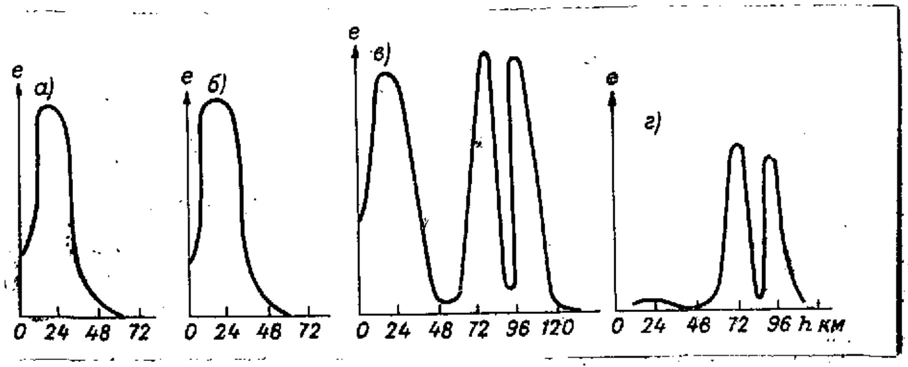

Fig. 5.10. Distribution of energy in height. Examples 24(a), 25(b), 26(c), 27(d).

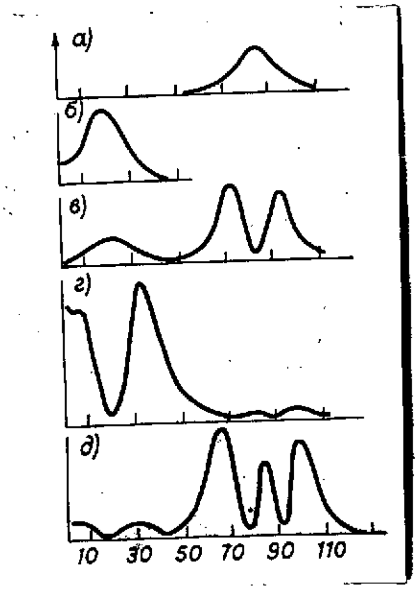

Fig. 5.11. Distribution of energy in height. Examples 19(a), 20(b), 21(c), 22(d), 23(e).

For comparison it is shown in Fig. 5.10c, what happens with oscillations of mode 1, when the characteristic curve withdraws to the left. We see, that the energy has begun passing into the upper wave guide. With further decrease of period it will completely pass into the upper wave guide, but it is not shown here. From Fig. 5.10d it can be seen, how the energy of mode 2 oscillations appeared prior to characteristic curve becoming a part of composite mode h $\approx$10 km. The energy was also concentrated in the upper wave guide. Therefore, with decrease of period, when this curve became a part of complex curve, energy passes into the lower wave guide.

Fig. 5.11 shows a series of curves, corresponding to examples 19-23. Here, the calculation is of waves of the same length, but different modes, starting from mode o and ending with mode 4. There is an interesting alternation of wave guides with the increasing number of mode the number of energy peaks, naturally, increases also.

The complex mode h 10 km, corresponding to waves with highest group velocity of propagation, plays, apparently, the most significant role in wave propagation from high intensity disturbances.

Wave guide properties of laminated atmosphere in respect of acoustical waves were investigated in the well known book of L.M. Brekhovskikh (1957), Yu. Gazaryan (1961) investigated them for acoustical gravitational waves, but his results are very sketchy. Once more we mention the work of Press and Harkrider and Pfeffer and Zarichny. Interesting observation material is given in the work of Diamond, (1963).

CHAPTER - 6.

EXPANSION ACCORDING NATURAL OSCILLATIONS

1. TWO-FOLD COMPLETENESS OF EIGEN FUNCTIONS:

In preceding Chapters determination was made of all the natural oscillations of atmosphere and some of their properties were investigated. Most frequently, these natural oscillations have interest not as themselves, but as those elementary solutions, which make up any transitory solutions of a system of equations in hydrodynamics. In this Chapter we shall deal with questions, related to the possibility of such expansion, with formulas, which permit the effective calculation of the factors of expansion. The most significant in this case will be the energy ratios, determined in the preceding Chapter..

We shall be interested now in motions of not very large scales of time and space in connection with the application of the theory (in the next Chapter) to the study of disturbance propagation from instanteneous point source. Because of this it is possible without any detriment to analyse the model of atmosphere above the non-rotatory earth. If in the main system (1.1') - (1.5') the Coriolis terms and discarded, it will be easy to see, that further elimination of unknown quantities and reduction of system to one equation can be carried out generally, not taking into account the dependence on the time of the exponential, i.e., to obtain equation for divergence containing time derivatives. The equation, as can be easily checked, will be as follows:

$$X_{ttt} - c^2 X_{ttzz} - \left(\frac{dc^2}{dz} - \varkappa g\right) X_{ttz} - c^2 \Delta X_{tt} - g\beta \Delta X = 0. \qquad (6\text{-}1)$$

The equation was found to be of the fourth order in time, in spite of the fact, that the initial system was of the fifth order. This equation, therefore, does not contain some of the system's resolutions, namely, the stationary, independent of time. At the initial moment four conditions should be preset. The initial conditions for the function X and its three time derivatives are expressed through starting conditions of the five initial fields u, v, w, p, ρ . Thus, for example,

$$X_t = -\frac{1}{\bar{\rho}} \Delta P - \left[\frac{1}{\bar{\rho}}(P_z + g\rho)\right]_z \qquad (6\text{-}2)$$

Derivatives X_{tt} and X_{ttt} are calculated from the general formula

$$\tilde{f}_2 = \Delta\left(c^2 \tilde{f}_1 - g\varphi\right) + \left[\left(c^2 \tilde{f}_1\right)_z - \beta \tilde{f}_1 - g\varphi_z\right]_z \qquad (6\text{-}3)$$

where substitution has to be made either of $\tilde{f}_1 = X$, $\tilde{f}_2 = X_{tt}$, $\varphi = w$, or $\tilde{f}_1 = X_t$, $\tilde{f}_2 = X_{ttt}$, $\varphi = \frac{1}{\bar{\rho}}(P_z + g\rho)$. Passing from function X by replacement of variables to function y, as shown in Chapter 1, we obtain also the initial conditions:

$$\begin{aligned} Y(\varphi, \theta, X, 0) &= Y^{(0)}(\varphi, \theta, X), \\ Y_t(\varphi, \theta, X, 0) &= Y^{(1)}(\varphi, \theta, X), \\ Y_{tt}(\varphi, \theta, X, 0) &= Y^{(2)}(\varphi, \theta, X), \\ Y_{ttt}(\varphi, \theta, X, 0) &= Y^{(3)}(\varphi, \theta, X). \end{aligned} \qquad (6\text{-}4)$$

Another result of the non-rotation of the Earth is the fact, that the sign is not included in the equations. If σ at a given h is the natural frequency, then also is $-\sigma$. We denote by $y(\sigma, h; x)$ the solution of equation (4-1) for the vertical component corresponding to parameters σ and h. Assuming n to be the number of the characteristic curve of Laplace's equation

$$h = \frac{\sigma^2 a^2}{gn\ (n+1)},$$

and j the number of characteristic curve of the equation (4-1), it is convenient for instance, to number the acoustical curves by positive numbers, and gravitational by negative, then $j = \pm 1, \pm 2, \ldots$ The intersection points of characteristic curves of equations for horizonta and vertical components will get double numbers n, j. The corresponding and h we denote by

$$\sigma_{n,\ j},\quad h_{n,\ j} = \frac{\sigma^2_{n,\ j} a^2}{gn\ (n+1)},$$

and the eigen function by $y_{n,j}(x)$. The the general solution should be

$$Y(\varphi,\ \theta,\ X,\ t) = \sum_{n,s,j} e^{is\varphi}\ P_n^{\ s}\ (\cos^{\theta})\ Y_{n,j}(x) \times$$

$$\times\ (a_{n,s,j} e^{i\sigma_{n,j} t} + b_{n,s,j} e^{-i\sigma_{n,\ j} t}), \qquad (6.5)$$

where the factors a and b should be determined from the initial conditions. For this we expand the initial functions according to spherical harmonics

$$Y^{(k)}(\varphi, \theta, X) = \sum_{n, s} Y^{(k)}_{n,s} e^{is\varphi} P_n^{|s|}(\cos\theta) \qquad (k = 0, 1, 2, 3),$$

after which it will be necessary to find the expansion

$$Y^{(o)}_{n,s}(X) = \sum_j C_{n,s,j} Y_{n,j}(X),$$

$$Y^{(1)}_{n,s}(X) = \sum_j d_{n,s,j} Y_{n,j}(X),$$

$$Y^{(2)}_{n,s}(X) = \sum_j (-\sigma^2_{n,j})\, C_{n,s,j} Y_{n,j}(X),$$

$$Y^{(3)}_{n,s}(X) = \sum_j (-\sigma^2_{n,j})\, d_{n,s,j} Y_{n,j}(X),$$

where

$$C_{n,sj} = a_{n,s,j} + b_{n,s,j},$$

$$d_{n,s,j} = i\,\sigma_{n,j}(a_{n,s,j} - b_{n,s,j}).$$

This system separates into two independent systems: with factors $C_{n,s,j}$

$$Y^{(o)}_{n,s}(X) = \sum_j C_{n,s,j} Y_{n,j}(X),$$

$$Y^{(2)}_{n,s}(X) = \sum_j (-\sigma^2_{n,j})\, C_{n,s,j} Y_{n,j}(X)$$

and with factors $d_{n,\ s,\ j}$

$$Y^{(1)}_{n,s}\ (X) = \sum_j\ d_{n,s,j}Y_{n,j}(X)$$

$$Y^{(2)}_{n,s}\ (X) = \sum\ (-\sigma^2_{n,j})\ d_{n,s,j}Y_{n,j}(X).$$

We shall take n and s as fixed, and shall omit these indices. The systems obtained are absolutely indentical. It is required to find factors C_j such, that

$$f\ (X) = \sum_j C_jY_j(X),$$

$$g\ (X) = \sum_j \sigma^2_{j,}\ C_jY_j(X). \tag{6-6}$$

where f and g are the known functions.

For the existence of such an expansion the significance, of course, is of the system's completeness of functions $Y_j(X)$. However, the simple completeness is not enough. We must find a concurrent expansion of two functions f and g with the same factors c_j. The possibility of this type of expansion for any f and g is known as the double complete system of functions $Y_j(X)$ (Keldysh, 1951).

The proof of the two-fold completeness and formulas for the factors will be given in the next paragraphs, but now we shall explain the concept of two-fold completeness in a simple particular case.

Assuming β = const, if we take instead of our more complex a simple boundary condition on the earth is surface, Y = 0, the matter is quite simple. We can prove in the equation.

$$Y'' + \left\{ -\frac{1}{4} - H^2k^2 + \frac{H}{xg}\left(\sigma^2 + \frac{\beta k^2 g}{\sigma^2}\right)\right\} Y = 0 \qquad (6\text{-}7)$$

($k^2 = n\ (n + 1)/a^2$) denote $\sigma^2 + \beta k^2 g/\sigma^2 = \mu^2$. Then there is a classical problem of Schturm-Liuville with natural parameter and linear boundary conditions. It is known that this problem has a complete system of eigen functions. But each μ^2 value is obtained at two different σ^2 values, since the μ^2 is expressed through σ^2 fractionally. (Strictly speaking, it is also necessary here to prove, that the minimum of function $\sigma^2 + \beta k^2 g/\sigma^2$ is less than the minimum eigen value μ^2, otherwise the least eigen values μ^2 will be obtained at complex σ^2 values, whereas, we are interested only in real values. The corresponding property is known as "high-intensity damping" and will be proved in the next paragraph). It is easy to see, that the system of eigen functions, composed of two similar sets, has a two-fold completeness in the sense of possible joint expansion of (6-6) for two arbitrary functions. Actually, by scalar multiplication of both the portions of equality (6-6) by one of the eigen functions y (x), on the strength of the latter's orthogonality, we get a system of two equations with two unknown factors C_j

$$(f,\ \psi_r) = C_r + C_{r'}\ ,$$

$$(g,\ \psi_r) = -\sigma_r^2 C_r - \sigma_r^{2}{}' C_{r'}\ ,$$

where σ_r^2, is another value of parameter σ^2, at which we obtain the same μ^2 and the same eigen function. For the existence of the required joint expansion it is necessary and sufficient, that all these systems had a solution. But this is actually so, because all the σ_j^2 are distinct one from another.

If σ^2 is included in the marginal condition, then the situation is considerably more complex. In this case (at β = const) the two-fold completeness is proved in the Author's work (1960). It should be mentioned, that here we do not get any more twice the same set of eigen functions, but two different complete sets, one of which corresponds to acoustical, and the other to gravitational waves. There is also no orthogonality here of eigen functions.

One more thing should be mentioned. Since with σ the $-\sigma$ is also an eigen value, the resolution of Koshi problem for the equation of fourth order did not require a four-fold completeness of eigen functions and it was possible to take twice one and the same two-fold complete set. If we take a model of the rotating earth, the matter here will be different: for positive and negative σ it would be necessary to take different characteristic curves of the equation for the horizontal component and, therefore, different eigen functions. Moreover, there would be an addition of one more set, related with gravitational gyroscopic waves, in complete accordance with the system having a fifth time order.

Paras 2-4 contain purely mathematical, slightly abstract and complex investigation of a two-fold completeness; in superficial reading these paragraphs may be omitted. The importance for applications is only of formulas for expansion factors of para 5 (6.49) - (6.50), which will be used in the next chapter.

2. PROOF OF COMPLETENESS. BOUNDARY CONDITION Y = O:

We begin by proving a two-fold completeness of the eigen functions system for a simpler problem with boundary condition on solid surface Y = O at X = O. The alternation, fixed in Chapter 4, of the eigen values of both the problems will make it possible for us in the next paragraphs to deduce hence the completeness of eigen function for our more complex problem.

To simplify the demonstration we shall assume the atmosphere as finite, by placing at a certain height X_o boundary condition independent of σ^2, for example Y =). We shall denote $\lambda = \sigma^2$. We shall reduce the problem now without difficulty to the form applicable in which are certain known theorems, which fix the completeness. Let us take the operator

$$F = -\frac{d^2}{dx^2} + \left(\frac{1}{4} + H^2k^2\right)$$

in condition Y = O at X = O and at $X = X_o$. This is a positively determined operator, having inverse [integral operator with core, which is the Green's function of equation $-y'' + \left(\frac{1}{4} + H^2k^2\right)y = 0$]. We denote this inverse operator F^{-1}. It is a positive, quite continuous operator. Therefore, there is operator $F^{-1/2}$. We substitute $Y = F^{-1/2}u$ in equation (6-7) and to both the portions of this equation we apply operator $F^{-1/2}$

$$u - \lambda \; F^{1/2} \frac{H}{xh} F^{-1/2} u - \frac{1}{\lambda} F^{-1/2} \; \frac{\beta Hk^2}{x} F^{-1/2} u = 0.$$

Here $H/\varkappa g$ and $\beta Hk^2/\varkappa$ - operators of multiplication by functions $H/\varkappa g$ and $\beta Hk^2/\varkappa$. The equation obtained is a particular case of equation

$$u = \lambda Gu + \lambda^{-1} Hu,$$

where u is vector of Gilbert's space and G and H, positive quite continuous operators. This type of equation was analysed in the work of N.G. Askerov, S.G. Krein and G.I. Laptev (1964); in a more suitable form for our object it is stated by G. Langer and M.G. Krein (1965), and also by M.G. Krein and I.C. Gokhberg (1965).

We shall demonstrate that the bunch of operators $I - \lambda G - \lambda^{-1} H$ (I - single operator) pertains in our case to highly significant particular type of bunches, known as highly damped. The bunch is denoted as highly damped, if for any vector u there is fulfilment of inequality

$$4(Gu, u)(Hu, u) < (u, u)^2.$$

In other words, equation

$$(u, u) - \lambda (Gu, u) - \frac{1}{\lambda} (Hu, u) = 0 \qquad (6\text{-}9)$$

as quadratic equation relatively to λ at any u has both the radicals real and different. The term "highly damped bunch" or "highly damped system" has its origin in mechanics, where these equations actually depict systems with high friction, ensuring purely aperiodic damping. In our case this term is only formal.

If we return from u to y, the equation (6-9) will be equivalent to equation

$$\int_0^{X_o} \left\{ y'' \bar{y} + \left[-\frac{1}{4} - H^2k^2 + \frac{H}{xg}\left(\lambda + \frac{\beta k^2 g}{\lambda}\right)\right] y\bar{y} \right\} dx = 0$$

for all functions y(x), which convert into zero at the ends of the interval. Assuming that at a certain y(x) this equation has immaterial radical $\lambda = \lambda_r + i\,\lambda_i$, $\lambda_i \neq 0$, then, by integrating this equation by parts, we get

$$- \int_0^{X_o} |y'|^2\, dx + \int_0^{X_o} \left[-\frac{1}{4} - H^2k^2 + \frac{H}{xg}\left(\lambda_r + \frac{\beta k^2 g \lambda_r}{|\lambda|^2}\right) |y|^2 dx \, + \right.$$

$$+ \, i\, \frac{\lambda_i}{xg} \int_0^{X_o} H\left(1 - \frac{\beta k^2 g}{|\lambda|^2}\right) |y|^2\, dx = 0. \tag{6-10}$$

The real and imaginary parts should individually be equal to zero

$$\int_0^{X_o} H\left(1 - \frac{\beta k^2 g}{|\lambda|^2}\right) |y|^2\, dx = 0 \tag{6-11}$$

and

$$\int_0^{X_o} |y'|^2 dx = \int_0^{X_o} \left[-\frac{1}{4} - H^2k^2 + \frac{H}{xg}\left(\lambda_r + \frac{\beta k^2 g \lambda_r}{|\lambda|^2}\right)\right] |y|^2 dx. \quad (6\text{-}12)$$

The same equalities hold true, if there is a real multiple radical. Now we bring into analysis energy integral. We denote $E = E_r + E_v$, where

$$E_r = k^2 \int_0^{X_o} \left| H \lambda y + g \left(y' - \frac{1}{2} y\right)\right|^2 dx, \quad (6\text{-}13)$$

$$E_B = \int_0^{X_o} \left| \lambda y' + \left(gk^2H - \frac{\lambda}{2}\right) y \right|^2 dx. \quad (6\text{-}14)$$

These formulas have the same appearance, as the formulas for horizontal and vertical energy in the preceding chapter, but this time they are written for complex λ and y functions. Besides, the latter are not assumed to be resolutions of equation. Next follows the conversion, similar to the one carried out in the proving of the theorem of virial

$$E_r = k^2 \int_0^{X_o} \Big\{ g^2 |y'|^2 + g\left(H\bar{\lambda} - \frac{g}{2}\right) y\bar{y} + g\left(H\lambda - \frac{g}{2}\right)\bar{y}'y + \\ + \left|H\lambda - \frac{g}{2}\right|^2 |y|^2 \Big\} dx.$$

Using for the substitution of the first term formula (6-12), replacing λ in the second and third term by $\lambda_r + i\,\lambda_i$ and integrating by parts the portion containing λ_r, we get.

$$E_r = k^2 \int_0^{X_o} H^2 \left\{ |\lambda|^2 + \lambda_r \left(\frac{g}{\varkappa H} + \frac{\beta g^2 k^2}{\varkappa H |\lambda|^2} - \frac{g H'_z}{H} - \frac{g}{H} \right) - g^2 k^2 \right\} |y|^2 dX + i\,\frac{\varkappa g^2 k^2 \lambda_i}{|\lambda|} \int_0^{X_o} H\,(\bar{y}'y - y'\bar{y})\,dX.$$

If we remember, that $\beta = (\varkappa - 1)g + \varkappa g H'_z$, the term in round brackets becomes considerably simpler and after certain calculations we shall get.

$$E_r = k^2 \int_0^{X_o} H^2 (|\lambda|^2 - g^2 k^2) \left(1 - \frac{\lambda_r \beta}{x^H |\lambda|^2} \right) |y|^2 dX + i\,\frac{\varkappa g^{2k2} \lambda_i}{|\lambda|} \int_0^{X_o} H\,(\bar{y}'y - y'\bar{y} \quad dX.$$

Exactly the same conversions we carry out in respect of E_v. Without bringing forward calculations, we write the result.

$$E_B = \int_0^{X_o} (g^2 k^2 - |\lambda|^2) \left(H^2 k^2 - \frac{\lambda_r H}{g} \right) |y|^2\,dX + i\,\frac{\varkappa g^2 k^2 \lambda_i}{|\lambda|} \int_0^{X_o} H\,(y'\bar{y} - \bar{y}'y)\;y\,dX.$$

By adding both the formulas we get.

$$E = \frac{\lambda_r |\lambda|^{-2}}{\varkappa\, g} \left(|\lambda|^2 - g^2k^2\right) \int_0^{X_o} H \left(|\lambda|^2 - \beta\, gk^2\right) |y|^2 dX. \quad (6\text{-}15)$$

The left portion is always positive. Therefore, the right portion also cannot be converted into zero, which contradicts assumption (6-11). Thus all the λ radicals are real and different.

It is already clear, that our problem or its equivalent problem (6.8) can have only real eigen values, since if the (6.8) is fulfilled, then it is even more so with (6.9). In Krein and Gokhberg book (p. 366-367) the following assertion is given: "For highly damped systems (6.8), where G and H are positive quite continuous operators, the theorem holds true of the two-fold completeness of the natural vectors system. Moreover, the system of natural vectors consists of two total bases. One of them corresponds to monotonously increasing succession of eigen values $\lambda_1^{(1)} < \lambda_2^{(1)} < \cdots$, for which the following relation is being fulfilled

$$\int_0^{X_o} H \left(\lambda^2 - \beta\, gk^2\right) |y|^2 dx > 0 \qquad (6\text{-}16)$$

(our denotations), i.e., they are the bigger radicals of the corresponding quadratic equations (6-10) (at $\lambda = \lambda_r$, $\lambda_i = 0$). The other corresponds to monotonously decreasing succession of eigen values $\lambda_i^{(2)} > \lambda_2^{(2)} > \lambda_3^{(2)} > \ldots$, for which

$$\int_0^{X_o} H \left(|\lambda|^2 - \beta\, gk^2\right) |y|^2 dx < 0, \qquad (6\text{-}16')$$

i.e., they are the least radicals of quadratic equations".

On the strength of relation (6-15) conditions (6-16) and (6-16') could be substituted by the following. For eigen functions of the first kind $\lambda > gk$, and of the second kind $\lambda < gk$. In other words, in the first case the frequencies are higher than Peckeris frequencies, corresponding to k, and in the second case lower. The first set pertains to acoustical waves, and the second to gravitational (somewhat conventionally, because so far we are analysing the simplified boundary conditions). The Peckeris frequencies divide the acoustical and the gravitational waves; the acoustical set of resolutions and the gravitational set form each separately total space basis.

This basis is not orthogonal. But it is the so called Riss basis (see Krein and Gohberg, p. 373), i.e., a basis obtainable from the ortho fixed by application of limited and reversible-limited operator.

3. <u>AUXILIARY FORMULAS AND ASYMPTOTIC EVALUATIONS:</u>

We change-over to proving the completeness of eigen functions at real marginal condition, dependent on eigen value. Let us recall the curves in chapter 4 of functions M(λ) and N(λ). They are reproduced in Fig: 6-1. At the intersection of both the curves we obtain natural frequencies. Frequencies of acoustical waves λ_α ($= \sigma_\alpha^2$) we numbered $\alpha = 1, 2, 3, \ldots$, and of gravitational waves $\alpha = -1, -2, -3, \ldots$ By λ_α^* we denoted natural frequencies of investigated problem with marginal condition y(0) = 0. As shown in chapter 4, eigen values of both the problems are intermittent, as shown in the figure. Thus, between the numbers λ_α and λ_α^* it is possible to fix reciprocally unambiguous correspondence, shown in the figure.

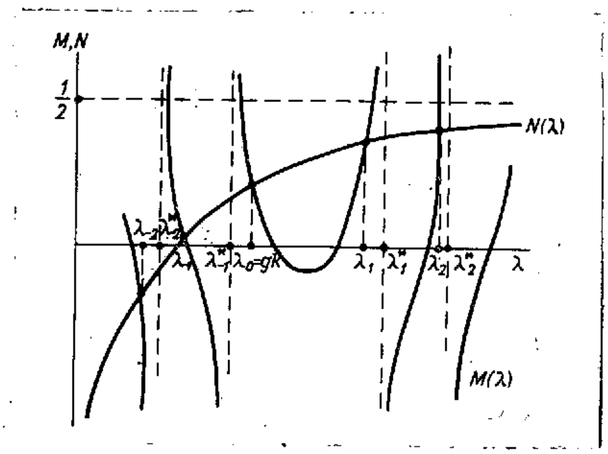

Fig: 6-1 - Eigen values of two marginal problems.

In the next paragraph it will be shown, how, using this correspondence, to re-expand functions, expanded by system $\{y_\alpha^*\}$, by system $\{y_\alpha\}$; thereby fixing the completeness of the latter. With this object we shall require primarily asymptotic evaluations of eigen values and eigen functions of high numbers, assuring convergence of those series, which will be encountered, and also some auxiliary functions.

At $\lambda \to \infty$, i.e., in the case of acoustical waves, the equation could be asymptotically written as

$$y'' + \frac{H}{\varkappa g} \lambda \ y = 0.$$

Substitution of $\xi = \int_0^X \sqrt[4]{H/\varkappa g}\,dx$, $y = \sqrt[4]{\varkappa g/H}\,\eta$ reduces this equation to $\eta'' + p\eta + \lambda\eta = 0$, where p is a certain function. At high λ this is equivalent to $\eta'' + \lambda\eta = 0$, whence we get

$$y \sim \sqrt[4]{\frac{\varkappa g}{H}} \sin\sqrt{\lambda} \int_x^{x_0} \sqrt[4]{\frac{H}{\varkappa g}}\, dx.$$

In the obvious way we obtain the asymptotic evaluations (letter K marks different constants)

$$\lambda^*_\alpha \sim \frac{\pi^2\alpha^2}{a^2}, \quad \left(a = \int_0^{X_0} \frac{H}{\varkappa g}\, dx\right), \quad \alpha = 1, 2, \ldots, \tag{6-17}$$

$$0 < \lambda^*_\alpha - \lambda_\alpha < \frac{\pi^2\alpha}{a^2} \tag{6-18}$$

$$K_1 \int_0^{X_0} (y^*)^2\, dx < K_2, \tag{6-19}$$

$$\int_0^{X_0} H\left[(\lambda^*_\alpha)^2 - \beta k^2 g\right] (y^*_\alpha)^2\, dx \sim K\,(\lambda^*_\alpha)^2, \tag{6-20}$$

$$y_\alpha(0) \sim K, \quad \left| y^{*\prime}_\alpha(0) \right| < K\alpha. \tag{6-21}$$

In the same way we obtain evaluations at $\lambda \rightarrow 0$ in the case of gravitational waves. The equation becomes simplified and appears as

$$y'' + \frac{H\beta\ k^2}{x}\ \frac{1}{\lambda}\ y = 0,$$

whence

$$y \sim \sqrt[4]{\frac{x}{H\ \beta\ \ k^2}}\ \sin\frac{1}{\sqrt{\lambda}} \int_{x}^{x_o} \sqrt{\frac{H\beta k^2}{x}}\ dx$$

and

$$\lambda^*_\alpha \sim \frac{b^2}{\pi^2\alpha^2},\ \left(b = \int_{o}^{x_o} \sqrt{\frac{H\ \beta\ k^2}{x}} dx\right),\ \alpha = -\ 1,\ -2,\ \ldots, \tag{6-22}$$

$$o < \lambda^*_\alpha - \lambda_\alpha < K\left(\lambda^*_\alpha\right)^2, \tag{6-23}$$

$$K_1 < \int_{o}^{x_o} (y^*_\alpha)^2\ dx\ < K_2, \tag{6-24}$$

$$\int_{o}^{x_o} H\left[(\lambda^*_\alpha)^2 - \beta\ k^2 g\right] (y^*_\alpha)^2\ dx \sim K, \tag{6-25}$$

$$y_\alpha\ (o) \sim \frac{K}{\alpha},\ \left| y^{*\prime}_\alpha\ (o)\right| < K\alpha\ . \tag{6-26}$$

We bring-in some integral relations. The eigen functions meet equations

$$y''_{\alpha} + \left\{ - \frac{1}{4} - H^2k^2 + \frac{H}{\chi g}\left(\lambda_{\alpha} + \frac{\beta k^2 g}{\lambda_{\alpha}}\right)\right\} y_{\alpha} = 0. \tag{6-27}$$

on condition

$$y'_{\alpha}(0) + \left[\frac{H(0)gk^2}{\lambda_{\alpha}} - \frac{1}{2}\right] y_{\alpha}(0) = 0 \tag{6-28}$$

and equations

$$y^{*''}_{\gamma} + \left\{ - \frac{1}{4} - H^2k^2 + \frac{H}{\chi g}\left(\lambda^*_{\gamma} + \frac{\beta k^2 g}{\lambda^*_{\gamma}}\right)\right\} y^*_{\gamma} = 0 \tag{6-29}$$

on condition

$$y^*_{\gamma}(0) = 0. \tag{6-30}$$

Multiplying (6-27) by $y^*_{\gamma}(x)$, and (6-29) by $y_{\alpha}(x)$, substracting from the first product the second and integrating with an estimate of boundary conditions, we get.

$$\frac{\lambda_{\alpha} - \lambda^*_{\gamma}}{\chi g \lambda_{\alpha} \lambda^*_{\gamma}} \int_0^{X_o} H\left(\lambda_{\alpha}\lambda^*_{\gamma} - \beta k^2 g\right) y_{\alpha} y^*_{\gamma}\, dx = y_{\alpha}(0)\, y^{*'}_{\gamma}(0) \tag{6-31}$$

In the same way, but using only (6-27) or (6-29), we shall have

$$\int_0^{X_o} H\left(\lambda^*_{\alpha}\lambda^*_{\gamma} - \beta k^2 g\right) y^*_{\alpha} y^*_{\gamma}\, dx = 0, \ \alpha \neq \gamma \tag{6-32}$$

and

$$\int_0^{X_o} H(\lambda_\alpha \lambda_\gamma - \beta k^2 g)\, y_\alpha\, y_\gamma\, dx = H(0) g^2 k^2 y_\alpha(0)\, y_\gamma(0), \alpha \neq \gamma . \quad (6\text{-}33)$$

Let us recall a formula, which was obtained somewhat more generally in the preceding paragraph

$$\int_0^{X_o} H \left[(\lambda_\alpha^*)^2 - \beta\, k^2 g \right] (y_\alpha^*)\, dx = \chi g \bar{E}_\alpha^* ,$$

$$\bar{E}_\alpha^* = \frac{\lambda_\alpha^* E_\alpha^*}{\left[(\lambda_\alpha^*)^2 - g^2 k^2 \right]}, \quad E_\alpha^* > 0. \quad (6\text{-}34)$$

4. <u>THE PROOF OF COMPLETENESS: THE REAL MARGINAL CONDITION:</u>

The two-fold completeness of the system of functions $\{y^*\}$ is fixed. Therefore, for any two functions f_1 and f_2 there exists an approximation

$$f_1 \sim \sum_{-N_1}^{N_1} C_\gamma^* y_\gamma^* ,$$

$$f_2 \sim - \sum_{-N_1}^{N_1} C_\gamma^* \lambda_\gamma^* \quad y_\gamma^* , \quad (6\text{-}35)$$

as accurate as required at sufficiently high N_1. In particular, taking as two such functions y_α and $-\lambda_\alpha y_\alpha$, we shall get

$$y_\alpha = \sum_{\gamma=-N}^{N} f_{\alpha\gamma} y^*_\gamma + r^{(N)}_\alpha,$$

$$-\lambda_\alpha y_\alpha = \sum_{\gamma=-N}^{N} f_{\alpha\gamma} \lambda^*_\gamma y^*_\gamma + s^{(N)}_\alpha, \qquad (\_-36)$$

where $r^{(N)}_\alpha$ and $s^{(N)}_\alpha$ are residual terms:

$$r^{(N)}_\alpha = \sum_{N+1}^{\infty} f_{\alpha\gamma} y^*_\gamma + \sum_{-\infty}^{-N-1} f_{\alpha\gamma} y^*_\gamma ,$$

$$s^{(N)}_\alpha = -\sum_{N+1}^{\infty} f_{\alpha\gamma} \lambda^*_\gamma y^*_\gamma - \sum_{-\infty}^{-N-1} f_{\alpha\gamma} \lambda^*_\gamma y^*_\gamma \qquad (6-37)$$

(In each of the written sums γ takes on all the indicated values, except $\gamma = 0$, to which none of the eigen functions correspond). Assuming we shall be able to find matrix $\| h^{(N)}_{\alpha\gamma} \|^N_{-N}$ inverse to matrix $\| f_{\alpha\gamma} \|^N_{-N}$. Then

$$y^*_\gamma = \sum_{\alpha=-N}^{N} h^{(N)}_{\gamma\alpha}\, y_\alpha - \sum_{\alpha=-N}^{N} h^{(N)}_{\gamma\alpha}\, r^{(N)}_\alpha,$$

$$-\lambda^*_\gamma\, y^*_\gamma = -\sum_{\alpha=-N}^{N} h^{(N)}_{\gamma\alpha}\, \lambda_\alpha\, y_\alpha - \sum_{\alpha=-N}^{N} h^{(N)}_{\gamma\alpha}\, s^{(N)}_\alpha. \qquad (6\text{-}38)$$

Further, the problem will be to prove, that in these formulas the residual terms (second terms of right-hand portions) could be made as low as desired at sufficiently high N. Then substituting hence the term for y^*_γ and $-\lambda^*_\gamma\, y^*_\gamma$ in (6-35), we shall get the following approximation:

$$f_1 \sim \sum_{\alpha=-N}^{N} c_\alpha\, y_\alpha,$$

$$f_2 \sim -\sum_{\alpha=-N}^{N} c_\alpha\, \lambda_\alpha\, y_\alpha, \qquad (6\text{-}39)$$

where the factors are

$$c_\alpha = \sum_{\gamma=-N_1}^{N_1} h^{(N)}_{\gamma\alpha}\, c^*_\gamma.$$

The accuracy could be made as high as desired, if first N_1 is made sufficiently high, and then N.

Thus, it is required: a) to find the factors $f_{\alpha\gamma}$ and to evaluate them, b) to evaluate the residual terms (6-37), c) to calculate the elements of the inverse matrix $h^{(N)}_{\alpha\gamma}$ and to evaluate them and d) to give evaluation to residual terms in formula (6-38), from which it would be evident, that at fixed γ they strive to zero at $N \to \infty$. Evaluations which should be carried out, involves rather bulky calculations. We shall omit the details, which, anyhow, it is not hard to restore every time, using evaluations of the preceding paragraph.

In order to find the factors $f_{\alpha\gamma}$, we multiply both the portions of the first of equations (6-36) by $H\beta k^2 g y^*_{\bar\gamma}$, of the second by $H\lambda^*_{\bar\gamma} y^*_{\bar\gamma}$, add and integrate by x ($\bar\gamma$ - certain fixed value of index γ). Formula (6-32) will give in this case

$$\int_0^{X_o} H\,(\lambda_\alpha \lambda^*_{\bar\gamma} - \beta k^2 g)\, y_\alpha\, y^*_{\bar\gamma}\, dx =$$

$$= f_{\alpha\bar\gamma} \int_0^{X_o} H\,(\lambda^*_{\bar\gamma} \lambda^*_{\bar\gamma} - \beta k^2 g)\, y^*_{\bar\gamma} y^*_{\bar\gamma} dx, \quad \alpha \neq \bar\gamma .$$

Recalling (6-31), it is possible to obtain

$$f_{\alpha\gamma} = \frac{\varkappa_g \lambda_\alpha \lambda^*_\gamma\, y_\alpha(0)\, y^{*\prime}_\gamma(0)}{(\lambda_\alpha - \lambda^*_\gamma) \int_0^{X_o} H\,(\lambda^*_\gamma \lambda^*_\gamma - \beta k^2 g)\, y^*_\gamma\, y^*_\gamma\, dx} \tag{6-40}$$

taking into account besides (6-34), we get

$$f_{\alpha\gamma} = \frac{\lambda_\alpha \lambda^*_\gamma\, y^{(0)}_\alpha\, y^{*\prime}_\gamma(0)}{(\lambda_\alpha - \lambda^*_\gamma)\, E^*_\gamma} \tag{6-41}$$

Formula (6-40) permits giving the following evaluation, if we also take into account (6-20), (6-21), (6-25) and (6-26)

$$
|f_{\alpha\gamma}| \quad
\begin{cases}
\dfrac{\alpha^2}{\gamma \cdot |\alpha^2 - \gamma^2|} & \text{at } \alpha > 0, \quad \gamma > 0 \\[2ex]
\dfrac{\gamma}{\alpha \cdot |\alpha^2 - \gamma^2|} & \text{at } \alpha < 0, \quad \gamma < 0 \\[2ex]
\dfrac{1}{\gamma} & \text{at } \alpha > 0, \quad \gamma < 0 \\[2ex]
\dfrac{1}{\alpha^3 \gamma^3} & \text{at } \alpha < 0 \quad \gamma > 0.
\end{cases}
\tag{6-42}
$$

Now we resolve problem "b" - evaluation of residual terms in (6-37). We shall evaluate integrals of the squares of these terms. Functions Y^*_γ $(\gamma > 0)$ are asymptotically orthogonal (with H weight), whereas, the integrals of their squares are confined within a constant range [formula (6-19)]; exactly the same is the case with $\gamma < 0$. Therefore, the square integral of residual terms gets evaluated by the squares sum of expansion factors by y^*_γ :

$$\int_0^{X_o} \left[r_\alpha^{(N)}\right]^2 dX \sim \sum_{\gamma=N+1} f_{\alpha\gamma}^2 + \sum_{\gamma=-\infty}^{-N-1} f_{\alpha\gamma}^2 ,$$

$$\int_0^{X_o} \left[s_\alpha^{(N)}\right]^2 dX \sim \sum_{\gamma=N+1}^{\infty} \left(\lambda_\gamma^*\right) f_{\alpha\gamma}^2 + \sum_{\gamma=-\infty}^{-N-1} \left(\lambda_\gamma^*\right)^2 f_{\alpha\gamma}^2 \qquad (6\text{-}43)$$

Let us evaluate, for instance, $s_\alpha^{(N)}$ at $\alpha > 0$

$$\int_0^{X_o} \left[s_\alpha^{(N)}\right]^2 dX \sim \sum_{\alpha=N+1}^{\infty} \frac{\gamma^4 \ \alpha^4}{\gamma^2(\alpha^2 - \gamma^2)^2} +$$

$$+ \sum_{\gamma=-\infty}^{-N-1} \frac{1}{\gamma^2} < K\alpha^4 \sum_{\gamma=N+1}^{\infty} \frac{\gamma^2}{(\alpha^2 - \gamma^2)^2} \sim K\alpha^4 \int_{N+1}^{\infty} \frac{x^2 dx}{(x^2 - \alpha^2)^2} =$$

$$= K\alpha^3 \int_{\frac{N+1}{\alpha}}^{\infty} \frac{x^2 dx}{(x^2 - 1)^2} < K\alpha^3 \left(\frac{N+1}{\alpha} - 1\right)^{-1} .$$

Similarly the evaluation is carried out of this residual term at $\alpha < 0$, as well as of the second residual term. Here is the result of evaluation.

$$\int_0^{X_0} \left[s_\alpha^{(N)}\right]^2 dx,\ \int_0^{X_0} \left[r_\alpha^{(N)}\right]^2 dx \leqslant \begin{cases} K_\alpha{}^3 \left(\dfrac{N+1}{\alpha} - 1\right)^{-1}, \alpha > 0 \\ K_\alpha^{-3} \left(\dfrac{N+1}{\alpha} - 1\right)^{-1}, \alpha < 0 \end{cases} \tag{6-44}$$

Now we proceed to resolving problem "c" - finding of inverse matrix. Using formula (6-41) we calculate the determinant of matrix $\| f_{\alpha\gamma} \|_{-N}^{N}$

$$\det f_{\alpha\gamma} = \frac{\prod\limits_{\alpha = -N}^{N} y_\alpha(0)\ y_\alpha^{*\prime}(0) \lambda_\alpha \lambda_\alpha^*}{\det \left(\lambda_\alpha - \lambda_\gamma^*\right)^{-1}}$$

The remaining determinant is calculated very simply see, [for instance, the well known book of Polia and Sega (1956)]. We have

$$\left|\det f_{\alpha\gamma}\right| = \frac{\prod\limits_{-N}^{N} y_\alpha(0)\ y_\alpha^{*\prime}(0) \lambda_\alpha \lambda_\alpha^*}{\bar{E}_\alpha^*\left(\prod\limits_{\substack{\alpha,\gamma \\ \gamma<\alpha}} (\lambda_\alpha^* - \lambda_\gamma^*)(\lambda_\alpha - \lambda_\gamma) \Big/ \prod\limits_{\alpha,\gamma} (\lambda_\alpha^* - \lambda_\gamma)\right)}.$$

The determinant is distinct from zero. In order to find the element $h^{(N)}_{\gamma\alpha}$, it is necessary to calculate the minor (i.e., determinant of the same type, composed of the same elements, but with omitted elements of the α - th line and γ-th column) and then to divide it by the whole determinant. We find.

$$\left|h^{(N)}_{\gamma\alpha}\right| = \left|\frac{E^*}{\lambda_\alpha \lambda^*_\gamma y_\alpha(0)\, y^*_\alpha(0)}\right| \cdot \left|\prod_{\substack{l=-N \\ l\neq\alpha,\gamma}}^{N} \frac{(\lambda_\alpha-\lambda^*_l)\;\lambda^*_\gamma-\lambda_l}{(\lambda_\alpha-\lambda_l)(\lambda^*_\gamma-\lambda^*_l)}\right| \times$$

$$\times \left|\frac{(\lambda_\alpha-\lambda^*_\gamma)(\lambda_\alpha-\lambda^*_\alpha)\;(\lambda_\gamma-\lambda^*_\gamma)}{(\lambda_\alpha-\lambda_\gamma)\;(\lambda^*_\alpha-\lambda^*_\gamma)}\right|, \qquad \alpha\neq\gamma .$$

Now we evaluate $h^{(N)}_{\gamma\alpha}$ at fixed γ by N and α. We take the multiple

$$\prod_{\substack{l=-N \\ l\neq\alpha,\gamma}}^{N} \left|\frac{(\lambda_\alpha-\lambda^*_l)\;(\lambda^*_\gamma-\lambda_l)}{(\lambda_\alpha-\lambda_l)\;(\lambda^*_\gamma-\lambda^*_l)}\right| = \prod_{\substack{l=-N \\ l\neq\alpha,\gamma}}^{N} \left|1+\frac{(\lambda_l-\lambda^*_l)\;(\lambda^*_\gamma-\lambda_\alpha)}{(\lambda_\alpha-\lambda_l)\;(\lambda^*_\gamma-\lambda^*_l)}\right| . \qquad (6\text{-}45)$$

We analyse first α negative. The product we divide into two parts - for positive and negative l. For the first of these parts the fraction is evaluated as $O(l^{-3})$ if we use the corresponding asymptotic formulas (6-17) - (6-26). Thus, this portion of the product is limited by the absolute constant. Limited in the same way is the second portion of product, for negative 1. In fact, for the negative 1 the evaluation is as follows:

$$\prod_{l=-\alpha+1}^{-1}\left(1+\frac{K^{2}}{\left|\alpha^{2}-l^{2}\right|l^{2}}\right)<\exp K\sum_{l=1}^{\alpha-1}\frac{\alpha^{2}}{\left|\alpha^{2}-l^{2}\right|l^{2}}<K.$$

Now let α be positive. The evaluation will be slightly more difficult. Part of the product, corresponding to negative 1, is limited, since it can be evaluated in the following way: $\prod_{-N}^{-1}\left[1+O(l^{-4})\right]<K.$ But with positive 1 we have

$$K\prod_{\substack{l=1\\ l\neq\alpha}}^{N}\left|1+\frac{\alpha^{2}}{l(l^{2}-\alpha^{2})}\right|\sim K\exp\sum_{\substack{l=1\\ l\neq\alpha}}^{N}\frac{\alpha^{2}}{l\,(l^{2}-\alpha^{2})}\sim$$

$$\sim K\exp\left[\ln\sqrt{1-\frac{\alpha^{2}}{l^{2}}}\,\Bigg|_{\alpha+1}^{N+1}+\right.$$

$$\left.+\ln\sqrt{\frac{\alpha^{2}}{l^{2}}-1}\,\Bigg|_{l}^{\alpha-1}\right]\sim K\,\frac{1}{\alpha}\sqrt{1-\frac{\alpha}{N+1}}.$$

Let us sum up. With negative α multiplier (6-45) is limited, with positive α it does not exceed $K_{\alpha}^{-1}\sqrt{1-\alpha/(N+1)}$. The remaining multipliers are evaluated quite simply. At $\alpha>0$ we have

$$\left|\frac{\lambda_\alpha - \lambda_\gamma^*}{\lambda_\alpha - \lambda_\gamma}\right| < K, \quad \left|\frac{\lambda_\alpha - \lambda_\alpha^*}{\lambda_\alpha^* - \lambda_\gamma^*}\right| < \frac{K}{\alpha},$$

whereas at $\alpha < 0$

$$\left|\frac{\lambda_\alpha - \lambda_\gamma^*}{\lambda_\alpha - \lambda_\gamma}\right| < K, \quad \left|\frac{\lambda_\alpha - \lambda_\alpha^*}{\lambda_\alpha^* - \lambda_\gamma^*}\right| < \frac{K}{\alpha^4}.$$

Finally, we estimate the remaining multiplier

$$\left|\frac{1}{\lambda_\alpha \, y_\alpha^{(0)}}\right| < \frac{K}{\alpha^2}$$

at $\alpha > 0$ and the multiplier

$$\left|\frac{1}{\lambda_\alpha \, y_\alpha^{(0)}}\right| < K\alpha^3$$

at $\alpha < 0$. Combining all evaluations, we get at $\alpha > 0$

$$\left|h_{\gamma\alpha}^{(N)}\right| < K\alpha^{-1} \sqrt{1 - \frac{\alpha}{N + 1}} \tag{6-46}$$

and at $\alpha < 0$

$$\left|h_{\gamma\alpha}^{(N)}\right| < K\alpha^{-1}. \tag{6-47}$$

It remains to resolve problem "d", i.e., to evaluate residual terms in formula (6-38). Since for $r_\alpha^{(N)}$ and for $S_\alpha^{(N)}$ there are similar evaluations, then, apparently, it is required to evaluate the sums.

$$\sum_{\alpha=1}^{N} h_{\gamma\alpha}^{(N)} r_\alpha^{(N)}, \qquad \sum_{\alpha=-N}^{-1} h_{\gamma\alpha}^{(N)} r_\alpha^{(N)},$$

For this object we use (6-44), (6-46), (6-47):

$$\int_0^{x_0} \left(\sum_{\alpha=1}^{N} h_{\gamma\alpha}^{(N)} r_\alpha^{(N)} \right)^2 dx \leqslant \sum_{\alpha=1}^{N} \left| h_{\gamma\alpha}^{(N)} \right| \sqrt{\int_0^{x_0} (r_\alpha^{(N)})^2 \, dx} \leqslant$$

$$\leqslant K \sum_{\alpha=1}^{N} \frac{1}{\alpha^4} \sqrt{1 - \frac{\alpha}{N+1}}\, \alpha^{3/2} \left(\frac{N+1}{\alpha} - 1 \right)^{-1/2} =$$

$$= \frac{K}{\sqrt{N+1}} \sum_{\alpha=1}^{N} \frac{1}{\alpha^2} < \frac{K}{\sqrt{N}}$$

$$\sqrt{\int_0^{x_0} \left(\sum_{\alpha=-N}^{-1} h_{\gamma\alpha}^{(N)} r_\alpha^{(N)} \right)^2 dx} \quad \sum_{\alpha=-N}^{-1} \left| h_{\gamma\alpha}^{(N)} \right| \sqrt{\int_0^{x_0} (r_\alpha^{(N)})^2 \, dx} \leqslant$$

$$\leqslant K \sum_{\alpha=1}^{N} \frac{1}{\alpha} \alpha^{-3/2} \left(\frac{N+1}{\alpha} - 1 \right)^{-1/2} \quad \frac{K}{\sqrt{N}}$$

The residual terms actually strive to zero, which proves the theorem.

5. EXPANSION FORMULAS:

Thus, we have proved the two-fold completeness of the system of functions, i.e., the possibility of approximating (6-39) for any wo functions at any preset accuracy. But so far it does not follow from anywhere, that any two functions could be expanded into series by y_α . In other words, would not it happen, that for rising the accuracy of the approximation it would be necessary each time to take absolutely new linear combinations in the right-hand portions of (6-39). Now we shall demonstrate, how to calculate effectively factors c_α . It will be found, that factors c_α are independent of N, but depend only on α , and this terminates the proving of expandability into series.

To calculate factors c_α , we multiply both the portions of the first of equations (6-39) by $H\beta k^2 g y_\gamma$, the second of these equations by $H\lambda_\gamma y_\gamma$, add them up and integrate by x. To sum obtained we add the first of the equations, taken at x = 0 and multiplied by $\varkappa H(0)g^2k^2y_\gamma(0)$. Considering the integral relation (6-33), we get

$$\int_0^{X_0} H(\beta k^2 g f_1 + \lambda_\gamma f_2) y_\gamma \, dx + \varkappa H(0)g^2k^2 f_1(0)y_\gamma(0) =$$

$$= \left[-\int_0^{X_0} H\left(\lambda_\gamma^2 - \beta k^2 g\right) y_\gamma^2 \, dx + \varkappa H(0)g^2k^2y_\gamma^2(0)\right] c_\gamma . \qquad (6\text{-}48)$$

This equation could be transformed, recalling the formula deduced in chapter 5, for the sum of horizontal and vertical energy.

$$E_r + E_B = \frac{E}{2} = \lambda^{-2}(\lambda^2 - g^2k^2)\left[\int_0^{X_o} H(\lambda^2 - \beta gk^2)\, y^2\, dx - \right.$$

$$\left. - \chi\, H(0) g^2 k^2 y^2(0)\right].$$

Then the formula for the factors could be rewritten in this way:

$$C_\gamma = \frac{\int_0^{X_o} H\, (\beta\, k^2 g f_1 + \lambda_\gamma\, f_2)\, y_\gamma\, dx + \chi\, H(0) g^2 k^2 f_1(0)\, y_\gamma\, (0)}{-\lambda_\gamma^2 \frac{E_\gamma}{2} \left(\lambda_\gamma^2 - g^2k^2\right)}. \qquad (6\text{-}49)$$

It would be expedient also to recall, that E_γ is actually energy with accuracy upto multiple. If the true energy value is denoted by $\mathcal{E}'_\gamma$ (see end para 1 of chapter 5), then we get

$$\mathcal{E}_\gamma = \frac{\chi\, \bar{P}o\, \sigma_\gamma^2}{2\left(\sigma_\gamma^4 - g^2k^2\right)^2}\, E_\gamma$$

and

$$C_\gamma = \frac{\chi\, po \left[\int_0^{X_o} H\left(\beta\, k^2 g f_1 + \sigma_\gamma^2 f_2\right) y_\gamma\, dx + \chi\, H(0) g^2 k^2 f_1(0)\, y_\gamma(0)\right.}{-\sigma_\gamma^2\left(\sigma_\gamma^4 - g^2k^2\right)\mathcal{E}_\gamma}. \qquad (6\text{-}50)$$

The factors, as stated, are independent of N.

We make one more remark. We analysed the problem for the sake of simplicity at the end segment $[0, X_o]$. In this case the λ value, equal to gk, i.e., the Peckeris value ceases to be the exact eigen value. There is appearance of λ_o, near, but not equal to gk. The system completeness of eigen functions takes place, as demonstrated, without the estimate of this eigen value, i.e., the latter seems to be unnecessary. But at the same time formula (6-48) gives for factor C_o, as well as for the other factors, value, generally speaking, distinct from zero. Is this not a contradiction?

The answer to this question is the fact, that in deduction of formula (6-48) we assumed not only the system completeness of the functions, but also that the arbitrary pairs of functions could be approximated not only in mean square, but also at $X = 0$. With this the system y_α at $\alpha \neq 0$ is found to be insufficient, and it is necessary to take into account function y_o also. The position here is quite similar to that, which takes place in the following example.

Let us take two systems of functions at segment $[0, \pi]$, namely, the system $\left\{ \sin \left(n - \frac{1}{2}\right) \left(\pi - x\right)\right\}$ and the system $\left\{\sin n(x - \pi)\right\}$, $n = 1, 2, \ldots$ Both the systems are complete in mean square. But any continuous function, equal to zero at $x = \pi$, could be expanded in uniformly converging series according the first system. But from the second system the expansion is only of those functions, which convert into zero also at $X = 0$. To make it possible from the second system to expand into uniformly converging series the same functions, as from the first, it should have an addition of one more function, for instance $\sin \frac{1}{2} (\pi - X)$.

Exactly the same position is in our case. For a uniform convergence, and therefore, also for the accuracy of formulas (6-48-6-50) it is necessary to estimate also y_o. But what will happen if we change over to the precise case of semi-straight line $[0, \infty]$? Here the position changes. Formula (6-49) at $\gamma = 0$ loses its meaning, there is an appearance in it of uncertainty, as $\lambda_o = gk$, $E_o = 0$. But it is possible to use the initial formula (6-48). The factor at C_o in the right portion is, generally speaking, distinct from zero; as regards the left portion, then it is equal to zero at arbitrary initial conditions. This at the beginning seems strange, since, into indicated formula enter arbitrary functions f_1 and f_2. Actually, these functions arbitrary are not but the initial fields u, v, w, p, ρ, through which these functions are expressed by formulas (6-2) and (6-3).

It f_1 and f_2, expressed by these formulas, are substituted in the left portion of (6-48) and pertinent damping is assumed of initial fields with height (finiteness of energy), we shall get equality to zero of this left portion, i.e., of factor C_o. Thus, actually, the Peckeris eigen function does not enter into expansion. If the semi-straight line $[0, \infty]$ is substituted by segment $[0, X_o]$, then it will enter into the expansion, but with very low weight, as these problems are similar.

This statement conforms to the remark made in Chapter 1, in regard to the fact that Peckeris waves carry infinite energy and, therefore, cannot enter into conposition of real solution with finite energy.

We must recall again, that if the solution is expanded in eigen values, then its corresponding energy is the sum of energies of the component oscillations, in accordance with additive energy, proved in Chapter 5.

CHAPTER - 7

PROPAGATION OF DISTURBANCE FROM INSTANTENEOUS POINT SOURCE:

(1) PROPAGATION VELOCITY OF DISTURBANCE:

In preceding chapters the investigation was of periodically time dependent solutions, having the shape of global waves. Thereby it was assumed, that from the excitation moment of this wave there is a sufficiently long time interval, so that the wave gets fixed.

The study of these waves is interesting, when we are dealing with sufficiently strong disturbances, as are able to encircle the globe several times without losing considerable part of their energy. This condition is fully met by the large scale waves, of the type of Rossbi waves, and also acoustical gravitational waves, evident as a result of particularly strong disturbances of the atmosphere, such, as for instance, the famous explosion of Karakatau volcano and the explosions in the tests of nuclear arms. Here, the waves are mainly of two-dimensional type, corresponding to h = 10 km, propagating without dispersion with velocity about 300 m/sec.

However, wave resolutions permit the solution of other class of problems also - problems of transient oscillations, since any transient resolution could be expanded from the wave resolutions. Local disturbance of the atmosphere excites a wide spectrum of waves. The resolution represents superposition of these waves. Hence, these waves diverge with different velocity due to dispersion. We know that phase and, specially, group velocities of acoustical waves are considerably higher than those gravitational of waves. Therefore, the acoustical waves, as may be expected, diverge considerably faster than the gravitational waves.

Sometimes after the passage of wave front there should remain mainly the gravitational component.

In the first paragraphs we shall analyse a model of the flat earth and the discussion will be confined to the simplest case, admitting analytical solution, - the case of isothermal stratification. Our first problem will be to find the solution in the case, when the initial disturbance is of point-type shaped, i.e., to find the function of Kosh problem effect. Hence, we will have to divide this resolution into parts, corresponding to acoustical and gravitational waves, and, finally, to investigate the time behavior of these parts, it is at $t \infty$. The asymptotic formulas obtained will actually show considerably higher damping of the acoustical part as compared with that of the gravitational.

The results will also be given of numerical calculations, carried out by Romanova (1966). It will be found that asymptotic formulas at $t \infty$ depict quite well (atleast qualitatively) resolutions for comparatively low t values. In the last paragraph we shall discuss the case of the spherical earth and the real stratification.

Let us take the system familiar to us, of equations in Cartesian coordinates

$$\frac{\partial u}{\partial t} = - \frac{1}{\bar{\rho}} \frac{\partial p}{\partial x} + lv,$$

$$\frac{\partial v}{\partial t} = - \frac{1}{\bar{\rho}} \frac{\partial p}{\partial y} - lu,$$

$$\lambda \frac{\partial w}{\partial t} = - \frac{1}{\bar{\rho}} \frac{\partial p}{\partial z} - g \frac{\rho}{\bar{\rho}},$$

$$\frac{\partial p}{\partial t} = - \bar{p} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - \frac{d\bar{\rho}}{dz} w,$$

$$\frac{\partial p}{\partial t} = - c^2 \bar{\rho} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + g\bar{p} w. \qquad (7.1)$$

Here λ - formal factor, equals to one. It is brought in in order to see, how the results change in quasistatic approximation ($\lambda = 0$). In this paragraph we shall assume $\lambda = 1$. We are dealing now with Kosh problem, i.e., assume, that at $t = 0$, the values of all the functions are known. We must first mention that the resolution of Kosh problem is unique and, moreover, it is possible to indicate the zone of initial data effect. We prove this by means of energy integrals. For each point of four-dimensional space (t, x, y, z) it is possible to plot a characteristic cone with apex at this point, resting on a certain closed zone of three-dimensional space (x, y, z) at $t = 0$. At each surface point of this cone the perpendicular meets the condition

$$n_t^2 - c^2 (n_x^2 + n_y^2 + n_z^2) = 0. \qquad (7.2)$$

Now we take the relation familiar to us for energy

$$\frac{\partial}{\partial t} \left\{ \bar{p} \; \frac{u^2 + v^2 + w^2}{2} + \frac{1}{2 \varkappa \bar{p}} \left[p^2 + \frac{g}{\beta} (p - c^2 \rho)^2 \right] \right\} +$$

$$+ \frac{\partial}{\partial x} (pu) + \frac{\partial}{\partial y} (pv) + \frac{\partial}{\partial z} (pw) = 0 \qquad (7.3)$$

We integrate this relation by volume, limited by the characteristic cone with the apex at point (t_o, x_o, y_o, z_o) and hyperplanes $t = 0$, and $t = t_o - \xi$. The left portion of relation (7.3) has the aspect of four-dimensional divergence, therefore, the integral on four-dimensional volume could be substituted by the integral on its three-dimensional surface. This integral on top base O

$$E_{\varepsilon} = \iiint\limits_{O_{\varepsilon}} \left\{ \bar{p} \, \frac{u^2 + v^2 + w^2}{2} + \frac{1}{2\chi\bar{\rho}} \left[p^2 + \frac{g}{\beta} (p - c^2 p)^2 \right] \right\} \times$$

$$\times \, d\chi dy \, dz, \quad t = t_o - \zeta,$$

integral on bottom base O_o

$$- E_o = - \iiint\limits_{O_o} \left\{ \bar{p} \, \frac{u^2 + v^2 + w^2}{2} + \frac{1}{2\chi\bar{p}} \left[p^2 + \right.\right.$$

$$\left.\left. + \frac{g}{\beta} (p - c^2_{\rho})^2 \right] \right\} \, d\chi dy dz, \quad t = 0$$

and on lateral surface σ

$$S = \iiint\limits_{\sigma} \left\{ \left[\bar{p} \, \frac{u^2 + v^2 + w^2}{2} + \frac{1}{2\chi\bar{p}} \left(p^2 + \frac{g}{\beta} (p - c^2_{\rho})^2 \right) \right] n_t + \right.$$

$$\left. + (pu) \, n_{\chi} + (pv) \, n_y + (pw) \, n_z \right\} d\sigma .$$

We evaluate the last three terms of the integrand function, using (7.2), inequality of Koshi-Bunyakovskii and the fact that the average geometrical of two quantities does not exceed their arithmetic average

$$\left| (pu) \, n_{\chi} + (pv) \, n_y + (pw) \, n_z \right| \leqslant$$

$$\leqslant \quad p\sqrt{u^2 + v^2 + w^2} \sqrt{n^2_{\chi} + n^2_y + n^2_z} = p\sqrt{u^2 + v^2 + w^2} \, \frac{n_t}{c} =$$

$$= \frac{p}{c\sqrt{\bar{p}}} \sqrt{\bar{p} \, (u^2 + v^2 + w^2)} \, n_t \leqslant \left[\frac{p^2}{2\rho c^2} + \bar{p} \, \frac{u^2 + v^2 + w^2}{2} \right] n_t =$$

$$= \left[\bar{p} \, \frac{u^2 + v^2 + w^2}{2} + \frac{1}{2\chi\bar{p}} \, p^2 \right] \, n_t .$$

But the first term of integrand function is positive, therefore, the whole integral is positive, S 0, since the element of volume d is positive. From

$$E_{\varepsilon} - E_{o} + S = 0$$

it follows that

$$0 \leqslant E_{\varepsilon} \leqslant E_{o}.$$

In other words, the energy in region O_{ε} is lower than the energy in region O_{o}. Thus, the energy may only issue from the characteristic cone, but not to flow-in. If it is assumed, that all initial data in the zone O_{o} were identically equal to zero, than also $E_{\varepsilon} = 0$, and all the functions u, v, w, p, $p-c^2\rho$ are equal to zero in zone O_{ε} , and due to arbitrary - also throughout the cone.

In the case of constant c, i.e., in the case of isothermy, the c represents maximum propagation velocity of disturbance, i.e.,velocity of disturbance front motion.

(2) FUNCTION OF KOSHI PROBLEM EFFECT:

We shall now assume c = const, and convert the system in the same way as in Monin and Obukhov work (1958). With this object we bring in new unknown quantities $\rho u = \varphi_x - \psi_y$, $\rho v = \varphi_y + \psi_x$, $\rho\omega = X$. Then it is easy to obtain for and equations

$$\frac{\partial \varphi}{\partial t} = - p + l\psi ,$$

$$\frac{\partial \psi}{\partial t} = - l\varphi .$$

If now P, ρ and ψ are now excluded, the following system will be obtained for X and φ :

$$\left(\frac{\partial^2}{\partial t^2} + 1^2\right)\varphi = (x - 1)\ gX + c^2 \frac{\partial x}{\partial z} + c^2\Delta\varphi\ , \left(\Delta = \frac{\partial^2}{\partial_x^2} + \frac{\partial^2}{\partial_y^2}\right),$$

$$\lambda \frac{\partial^2 x}{\partial t^2} = \frac{\partial}{z}\left[(x - 1)\ gX + c^2 \frac{\partial x}{\partial z} + c^2\Delta\varphi\right] + g\left[\frac{\partial x}{\partial z} + \Delta\varphi\right]. \quad (7.4)$$

In the starting unknown quantities at t = 0, for initial system were ψ_o, φ_o, P_o, X_o, ρ_o, then for (7.4) the starting data will be

$$\varphi = \varphi_o, \frac{\partial\varphi}{\partial t} = 1\psi_o - P_o, \quad X = X_o, \quad \lambda\frac{\partial x}{\partial t} = -\left(\frac{\partial P_o}{\partial z} + gp_o\right). \quad (7.5)$$

System (7.4) has not the fifth, but the fourth time order. By solving it we find the remaining unknown quantities by means of one more time integration. Since the order of the system has decreased, then it naturally, has less solutions. Namely, if the starting data of initial system satisfied condition

$$\varphi_o \quad 1\psi_o - P_o \quad X_o \quad \frac{\partial P_o}{\partial z} + g\rho_o \quad 0, \qquad (7.6)$$

then for the new system (7.4) there will be zero starting conditions, and the solution will be identically equal to zero. However, the whole solution cannot be zero. It will be stationary, independent of time, and conditions (7.6) will be implementing identically (this is the condition of horizontal motion, solenoidality, geostrophicity and condition of quasi-statics). Stationary solutions correspond fully in the model of flat earth to inertia-gyroscopic waves, known to us.

At $\lambda = 0$, i.e., in condition of quasistatics, in the left portion of the second (7.4) equation instead of $\partial^2_X/\partial t^2$ will be found zero.

The time order of the system will decrease to the second.

Now we exclude from the system the unknown φ. We shall get

$$\left(\frac{\partial^2}{\partial t^2} + 1^2\right)\left(c^2 X_{zz} + xgX_z - X_{tt}\right) + \Delta\left[(x - 1)\, g^2 X + \lambda c^2 X_{tt}\right] = 0. \tag{7.7}$$

In this case the starting conditions will be the following

$$X = X_o,\ X_t = X_1,\ X_{tt} = X_2,\ X_{ttt} = X_3 \qquad t = 0, \tag{7.8}$$

where

$$\lambda X_1 = -\left(\frac{\partial p_o}{\partial z} + gp_o\right),\quad \lambda X_2 = \left(g + c^2 \frac{\partial}{\partial z}\right)\Delta\varphi_o + $$

$$+ \frac{\partial}{\partial z}\left(xg + c^2 \frac{\partial}{\partial z}\right) X_o,$$

$$\lambda X_3 = \left(g + c^2 \frac{\partial}{\partial z}\right)\Delta\,(1\psi_o - P_o) + \frac{\partial}{\partial z}\left(xg + c^2 \frac{\partial}{\partial z}\right) X_1.$$

After X will be found as a result of solving equation (7.7) with the starting conditions (7.8), it will be possible to find , by solving the first of (7.4) equations - Klein-Gordon type of equation. It should be mentioned, that a case is possible, when $X \equiv 0$, and $\varphi \not\equiv 0$. This will take place, if at the starting moment $X_o \equiv 0$, $\frac{\partial p_o}{\partial z} + g\rho_o \equiv$ $\equiv 0$, i.e., there is no vertical velocity and the fulfilment is of static equilibrium condition, and the φ_o and $1\psi_o - P_o$ are distinct from zero, but quite definitely depend on height as e^{-gz/c^2}. Then also

$$\varphi(X, Y, Z, t) = e^{-gz/c^2}\, \hat{\varphi}\,(X, Y, t),$$

where function $\hat{\varphi}$ satisfies the equation

$$\left(\frac{\partial^2}{\partial t^2} + 1^2 - c^2\Delta\right)\widehat{\varphi} = 0.$$

This is a two-dimensional solution, corresponding to Lamb's waves.

Now we shall speak only about the solution of equation (7.7) for function X. It does not depict anymore either the inertia-gyroscopic wave, or the Lamb's waves. We carry out the usual substitution of variables, which destroys the term with the first derivative

$$X = e^{-\frac{xg}{2c^2}z}\eta ,$$

after which the equation will be

$$\left(\frac{\partial^2}{\partial t^2} + 1^2\right)\left(c^2\eta_{zz} - \frac{x^2g^2}{4c^2}\eta - \lambda\eta_{tt}\right) +$$

$$+\Delta\left[(x-1)\, g^2\eta + \lambda c^2\eta_{tt}\right] = 0. \qquad (7.9)$$

The marginal conditions are as follows: on the earth's surface, at $z = 0$, it is assumed $\eta = 0$, and at infinity the condition is taken of η limiting. The starting conditions at $t = 0$, are the following:

$$\eta = \eta_o\left(= e^{\frac{xg}{2c^2}z}X_0\right),\quad \eta_t = \eta_1\left(= e^{\frac{xg}{2c^2}z}X_1\right),$$

$$\eta_{tt} = \eta_2\left(= e^{\frac{xg}{2c^2}z}X_2\right),\quad {}_{ttt} = \eta_3\left(= e^{\frac{xg}{2c^2}z}X_3\right).$$

Now we have a differential equation with constant factors and the starting conditions. The problem could be solved by means of Furier's conversion from variables x, y and Laplace's conversion by t, since the solution is being sought for on the semiaxis $t \geqslant 0$ at starting conditions. First we shall carry out Furier's conversion of the sought for function and of the starting conditions:

$$\eta(x, y, z, t) = \frac{1}{4\pi^2} \iint e^{i(k_1 x + k_2 y)} \bar{\eta}(k_1, k_2, x, t)\, dk_1\, dk_2,$$

$$\eta_a(x, y, z, t) = \frac{1}{4\pi^2} \iint e^{i(k_1 x + k_2 y)} \bar{\eta}_a x(k_1, k_2, x, t)\, dk_1 dk_2.$$

The equation will become

$$\left(1^2 + \frac{\partial^2}{\partial t^2}\right)\left(\bar{\eta}_{zz} - \frac{x^2 g^2}{4c^4}\bar{\eta} - \frac{\lambda}{c^2}\bar{\eta}_{tt}\right) -$$

$$- k^2\left[\frac{(x-1)\, g^2}{c^2}\bar{\eta} + \lambda\bar{\eta}_{tt}\right] = 0. \qquad (7.10)$$

After this we shall carry out Laplace's conversion by t. For this both the portions of equation (7.10) we multiply by e^{-pt} and integrate by t from 0 to , denoting,

$$\tilde{\eta}(k_1, k_2, z, p) = \int_0^\infty e^{-pt}\bar{\eta}(k_1, k_2, z, t)\, dt.$$

Then we shall get

$$(1^2 + p^2)\left(\tilde{\eta}_{zz} - \frac{x^2 g^2}{4c^4}\tilde{\eta} - \frac{\lambda p^2}{c^2}\tilde{\eta}\right) - k^2\left[\frac{(x-1)\, g^2}{c^2} + \lambda p^2\tilde{\eta}\right] = r, \qquad (7.11)$$

where the right portion r emerges with integration by parts from initial conditions

$$r = (\bar{\eta}_1)_{zz} + p(\bar{\eta}_0)_{zz} - \left(\frac{x^2 g^2}{4c^4} + \lambda k^2 + \frac{\lambda 1^2}{c^2}\right)(\bar{\eta}_1 + P\bar{\eta}_0) -$$

$$- \frac{\lambda}{c^2}(\bar{\eta}_3 + P\bar{\eta}_2 + P^2\bar{\eta}_1 + P^3\bar{\eta}_0).$$

Marginal conditions at z = 0 and z = remain the same as for . By resolving linear differential equation with invariable factors and estimate of marginal condition at lower boundary we will have

$$\tilde{\eta} = \int_0^z \frac{\operatorname{sh} R_p(k)\,(z - z_1)\, r}{R_p(k)\,(p^2 + 1^2)}\, dz_1 + A\,(p) \frac{\operatorname{sh} R_p\,(k)}{R_p(k)},$$

where

$$R_p(k) = \sqrt{\frac{\frac{x^2 g^2}{4C^4} + \frac{\lambda p^2}{C^2} + k^2 \left(\frac{(x-1)\, g^2}{C^2} + \lambda p^2\right)}{(1^2 + p^2)}}$$

The constant A(p) is determined from condition of limiting. It is not difficult in this case to obtain the following. If function r is projected oddly on the negative semiaxis, i.e., to determine r for negative z as

$$r\,(k_1,\, k_2,\, -z,\, p) = -\, r\,(k_1,\, k_2,\, z,\, p),$$

function $\tilde{\eta}$ will be

$$\tilde{\eta} = -\,\frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-R_p \quad z\, -\, z_1}}{(p^2 + 1^2)\, R_p}\, d\, z_1. \tag{7.12}$$

It remains now to revert to the Furier's and Laplace's conversions and to return to the previous function . First we reverse Furier's conversion. We donote

$$*\,(X,\, Y,\, z,\, p) = \quad_0 \quad e^{-pt} \quad (X,\, Y,\, z,\, t)\, dt.$$

We multiply both the portions of (7.12) by $\frac{1}{4\pi^2} e^{i\,(k_1 x + k_2 y)}$ and integrate by k^1 and k^2 from $-\infty$ to ∞. From the theorem of convolution (Furier's conversion of the product of two functions is the convolution of the Furier's conversions of these functions) it is possible to write

$$\eta^* = - \iiint_{-\infty}^{\infty} G^* (x - x_1, y - y_1, z - z_1, p)\, r^* (x_1\, y_1, z_1, p) \times$$

$$\times\, dx_1\, dy_1\, dz_1,$$

where

$$r^* (x, y, z, p) = (\eta_1)_{zz} + p\, (\eta_o)_{zz} -$$

$$- \left(\frac{x^2 g^2}{4c^4} + \lambda k^2 + \frac{\lambda 1^2}{c^2} \right) (\eta_1 + P\eta_o) - \frac{\lambda r}{c^2} (\eta_3 + p\eta_2 + p^2 \eta_1 + p^3 \eta_o)$$

and

$$G^* (x, y, z, p) = - \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} e^{i\,(k_1 x + k_2 y)}\, \frac{e^{-R_p |z|}}{2\,(p^2 + 1^2)\, R_p}\, dk_1 dk_2.$$

The last function could be calculated using formula

$$\iint_{-\infty}^{\infty} e^{i\,(k_1 x + k_2 y)}\, \frac{e^{-\sqrt{a^2 + (k_1^2 + k_2^2)\, b^2}}}{\sqrt{a^2 + (k_1^2 + k_2^2)\, b^2}}\, dk_1 dk_2 =$$

$$= 2\, \frac{e^{-\frac{a}{b}\sqrt{b^2 + (x^2 + y^2)}}}{b\sqrt{b^2 + (x^2 + y^2)}}$$

Then we shall have

$$G^* (x, y, z, p) = - \frac{1}{4\pi}\, \frac{\exp\left[-c^{-1}\sqrt{\frac{x^2 g^2}{4c^2} + \lambda p^2}\;\; S \right]}{\left[(x - 1)\, g^2 c^{-2} + \lambda p^2 \right] S}$$

where

$$S = \sqrt{z^2 + \frac{(1^2 + p^2)\, (x^2 + y^2}{(x - 1)\, g^2 c^{-2} + \lambda\, p^2}}$$

Now we have to invert Laplace's conversion. Assuming G (x, y, z, t) prototype of G* (x, y, z, p), i.e.,

$$G(x, y, z, t) = -\frac{1}{8\pi^2 i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} \frac{\exp\left[-\frac{1}{C}\sqrt{\frac{x^2 g^2}{4C^2} + \lambda p^2}\; s\right]}{\left[(x-1)\, g^2 C^{-2} + \lambda p^2\right] s}\, dp, \tag{7.13}$$

where $\gamma > 0$. In other words, the integration is carried out along a vertical straight in the right-hand semiplane of composite variable p. We denote also by G the integral operator with core $G(x - x_1, y - y_1, z - z_1, t)$, i.e.,

$$Gu = \iiint G(x - x_1, y - y_1, z - z_1, t)\, u(x_1, y_1, z_1)\, dx_1 dy_1 dz_1$$

Then the resolution is given by the formula

$$\eta = G\left(\Delta_3 - \frac{x^2 g^2}{4C^4} - \frac{\lambda 1^2}{C^2}\right)\eta_1 + \frac{\partial}{\partial t} G\left(\Delta_3 - \frac{x^2 g^2}{4C^4} - \frac{\lambda 1^2}{C^2}\right)\eta_0 - \frac{\lambda}{C^2}\left(G\eta_3 + \frac{\partial}{\partial t} G\eta_2 + \frac{\partial^2}{\partial t^2} G\eta_1 + \frac{\partial^3}{\partial t^3} G\eta_0\right), \quad \Delta_3 = \lambda\Delta + \frac{\partial^2}{\partial z^2}. \tag{7.14}$$

Here, all functions η_x are taken as extended by z unevenly on negative semiaxis.

In a particular case, when the starting conditions are preset so that only η_3 is distinct from zero, formula (7.14) is quite simple: $= C^{-2} G\eta_3$. Hence, it is clear that G (x, y, z, t) is the resolution of equation with δ-like starting condition, i.e., function of initial data effect.

(3) ACOUSTICAL AND GRAVITATIONAL PARTS OF RESOLUTIONS:

We have to investigate the effect function (7.13). We shall write it in a more convenient form

$$G(x, y, z, t) =$$

$$= -\frac{1}{8\pi^2 i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}\,\frac{\exp\left[-c^{-1}\sqrt{x^2 + y^2 + \lambda z^2}\,\sqrt{\dfrac{(\tau_1^2 + \lambda p^2)(\tau_3^2 + p^2)}{\tau_2^2 + \lambda p^2}}\right]}{\sqrt{(\tau_2^2 + \lambda p^2)(\tau_3^2 + p^2)}\sqrt{x^2 + y^2 + \lambda z^2}}\,dp, \qquad (7.15)$$

where

$$\tau_1 = \frac{xg}{2c\sqrt{\lambda}}, \quad \tau_2 = \sqrt{\frac{x - 1}{\lambda}}\,\frac{g}{c},$$

$$\tau_3 = \sqrt{\frac{l^2 (x^2 + y^2) + (x - 1)\, g^2 c^{-2} z^2}{x^2 + y^2 + \lambda z^2}},$$

$p = \pm i\tau_1, \pm i\tau_2, \pm i\tau_3$ - branching points of integrand function. First of all we shall try by deforming the contour of integration to convert the integral into a real one. With this object we have to imagine the behavior of integrand function in the complex plane. We select those branches of radicals in the formula of this function, which for real p values are positive. In the vicinity of infinitely removed point the function is unambiguous and behaves as

$$\frac{e^{p\left(t - c^{-1}\sqrt{x^2 + y^2 + \lambda z^2}\right)}}{p^2\sqrt{x^2 + y^2 + \lambda z^2}}.$$

Assuming $\lambda = 1$. At $t \leqslant C^{-1}\sqrt{x^2 + y^2 + z^2}$ the integrand function dampens, when Re $p \longrightarrow +\infty$, and at $t \geqslant C^{-1}\sqrt{x^2 + y^2 + z^2}$ – when Re $p \longrightarrow -\infty$. In the first case it is possible to shift the contour into $+\infty$, i.e., to drive γ to $+\infty$. Since in the right-hand semiplane there are no special points of integrand function, the integral is found to be zero. Thus, at $\sqrt{x^2 + y^2 + z^2} \geqslant Ct$ the function of effect is zero — a fact, that we know even before (para 1).

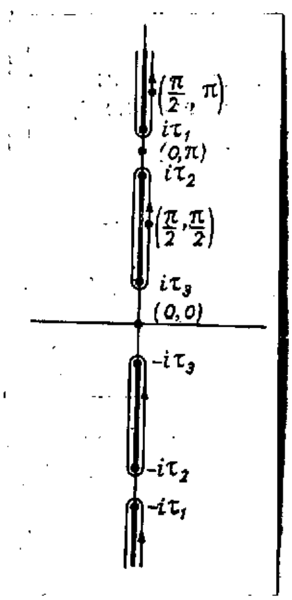

Fig.7.4 - Deformation of integration contour.

At $\sqrt{x^2 + y^2 + z^2} \leqslant Ct$ the integrand function dampens now in the left semiplane. But the contour cannot already be shifted into – , since, there are in its way special points of integrand function - the points of branching. Figure 7.1 shows these branching points on plane p. We draw the cuts, as shown in figure 7.1 by thick lines. Outside these cuts all radicals are unambiguous functions. The figure at some points shows in round brackets two numbers. Those are arguments of two radicals

$$\sqrt{\frac{(\tau_1^2 + p^2)\,(\tau_3^2 + p^2)}{\tau_3^2 + p^2}} \text{ and}$$

$$\sqrt{(\tau_2^2 + p^2)\,(\tau_3^2 + p^2)}$$

To the integral in (7.15) we add an integral from the same function, but taken on a straight line in the left semiplane from $-\gamma + i\infty$ to $\gamma - i\infty$. This integral will not change anything, as it is equal to zero - the integrand function dampens at infinity in the left semiplane. But the sum of two integrals is an integral on complex contour, shown in the figure. We break this integral into two parts - integral along the top loop, passing around the point $i\tau_1$, and also along the conjugated loop around $-i\tau_1$, and integral along the closed loop around $i\tau_2$ and $i\tau_3$ plus integral along the conjugated loop around $-i\tau_2$ and $-i\tau_3$. Analysing the signs of radicals at both the ends of sections, it is possible to obtain the following formulas. For the first part, which we shall name G_{ak} (due to causes which will become clear later), will be

$$G_{ak} = \frac{1}{2\pi^2} \times$$

$$\times \int_{\tau_1}^{\infty} \sin \gamma t \, \frac{\sin\left[c^{-1}\sqrt{x^2 + y^2 + \lambda z^2}\,\sqrt{\frac{(\tau^2 - \tau_1^2)\,(\tau^2 - \tau_3^2)}{\tau^2 - \tau_3^2}}\right]}{\sqrt{x^2 + y^2 + \lambda z^2}\,\sqrt{(\tau^2 - \tau_2^2)\,(\tau^2 - \tau_3^2)}}\, d\tau,$$

(7.16)

for the second

$$G_{rpab} = -\frac{1}{2\pi^2}\,\chi$$

$$\chi \int_{\tau_3} \sin \tau t \; \frac{\cos\left[c^{-1}\sqrt{x^2+y^2+\lambda z^2}\sqrt{\dfrac{(\tau_1^2-\tau^2)(\tau^2-\tau_3^2)}{(\tau_2^2-\tau^2)}}\right]}{\sqrt{x^2+y^2+\lambda z^2}\;\sqrt{\lambda(\tau_2^2-\tau^2)(\tau^2-\tau_3^2)}}\,d\tau .$$

Now we shall explain the physical meaning of dividing the function of effect into two parts and the meaning of denoting these parts. In chapter 4 during the investigation of waves in isothermal atmosphere we recognized acoustical and gravitational waves from their ultimate behavior in two extreme cases - with transition to [illegible] incompressibility, i.e., at $\chi \longrightarrow \infty$ and with transition to indifferent static balance, i.e., at $\chi \longrightarrow 1$. Let us see now, what happens to functions G_{ac} and G_{grav} in these ultimate transitions. We assume $\lambda = 1$.

Let first of all $\chi \longrightarrow \infty$. Then $\tau_1 = \sqrt{\chi g/4H} \longrightarrow \infty$. Hence, it already follows, that integral G_{ac} completely disappears. As regards the second integral G_{grav}, it, as can be easily checked, converts into

$$G_{rpab} = \frac{1}{2\pi^2}\,\chi$$

$$\chi \int_{\tau_3}^{\tau_2} \sin \tau t \; \frac{\cos\left[(2H)^{-1}\sqrt{x^2+y^2+z^2}\sqrt{\dfrac{\tau^2-\tau_3^2}{\tau_2^2-\tau^2}}\right]}{\sqrt{x^2+y^2+z^2}\;\sqrt{(\tau_2^2-\tau^2)(\tau^2-\tau_3^2)}}\,d\tau ,$$

i.e., undergoes only some not very considerable quantitative changes.

Let $\chi \longrightarrow 1$, in this case we shall assume $1 \longrightarrow 0$. Then, contrary-wise, G_{ac} changes little

$$G_{ak} = \frac{1}{2\pi^2} \int_{\tau 1}^{\infty} \operatorname{Sin} \tau t \frac{\sin\left[C^{-1} \cdot \sqrt{x^2+y^2+z^2} \sqrt{\tau 2 - \tau_1^2} \right.}{\sqrt{x^2+y^2+z^2} \quad \tau^2} d\tau ,$$

and

$$G_{rpab} = - \frac{t}{2\pi} \frac{\cos\left[(2H)^{-1} \sqrt{x^2+y^2+z^2} \right]}{\sqrt{x^2+y^2+z^2}} .$$

This function depends on time aperiodically.

These properties of function G_{ac} and G_{grav} give grounds to see in them actually the acoustical and gravitational portions of the function of effect G, all the more so, as even in the shape the integrals are written in a way, that G_{ac} represents superposition of harmonics sin t with frequencies higher than $_1 = \sqrt{\chi g/4H}$, and G_{grav}- with frequencies lower than $\tau_2 = \sqrt{(\chi - 1)g/\chi H}$, as it shoudl be for acoustical and gravitational wave. Thus G_{ac} in the form of superposition of acoustical and G_{grav} - of gravitational wave.

Finally we shall use parameter λ and see into what our functions of effect get converted in quasistatic approximation. At $\lambda \to 0$ out of three special points two ($i\tau_1$ and $i\tau_2$) withdraw into infinity; G_{grav} disappears in the same way as at $\chi \to \infty$, which quite corresponds to the known fact, that acoustical wave are absent in the approximation of quasistatics. Formula (7.15) becomes

$$G = - \frac{1}{8\pi 2^i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} \frac{\exp\left[-\tau_1 \tau_2^{-1} C^{-1} \sqrt{x^2+y^2} \sqrt{\tau_3^2 + p^2} \right.}{\tau_2 \sqrt{x^2+y^2} \sqrt{\tau_3^2 + p^2}} dp.$$

The prototype of Laplace's conversions we find in the tables of Laplace's conversions. It is equal to

$$G = -\frac{1}{4\pi}\sqrt{\frac{xH}{(x-1)\,g\,(x^2+y^2)}}\times$$

$$\times\begin{cases} J_o\left(\tau_3\sqrt{t^2 - \frac{x}{4\,(x-1)\,g\,H}\,(x^2+y^2)}\right), & t > \frac{1}{2}\sqrt{\frac{x\,(x^2+y^2)}{(x-1)\,gH}} \\ 0 & , t \leqslant \frac{1}{2}\sqrt{\frac{x\,(x^2+y^2)}{(x-1)\,gH}}. \end{cases}$$

The vertical propagation velocity of disturbances has become infinite, and the horizontal is equal to

$$\frac{2\sqrt{(x-1)\,gH}}{x} = \frac{2\sqrt{x-1}}{xC}$$

which constitutes $2\sqrt{\chi - 1}/\chi \approx 0.9$ of the previous velocity. Hence the $\lambda = 1$ will be assumed universally.

(4) ASYMPTOTIC BEHAVIOR OF EFFECT FUNCTION AT HIGH t VALUES:

The propagation nature of acoustical and gravitational waves is quite different as we know. Becuase of this it may be expected that the behavior also of both portions of the function of effect G_{ac} and G_{grav} will be radically different. A considerably higher damping should be expected of the acoustical portion in time and considerably less diffusivity of the wave front. Moreover, gravitational portion should display greater anisotropy - non-equality of horizontal and vertical directions should be shown more clearly.

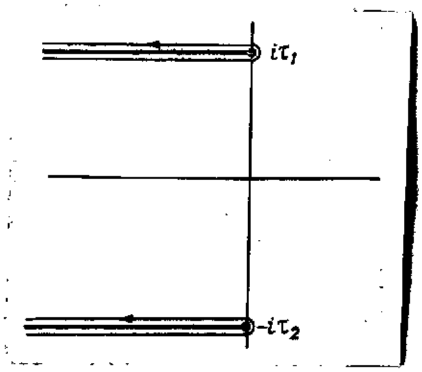

Fig.7.2 - Integration contour for asymptotic evaluation of G_{ac}.

Direct investigation of the integral concepts for G_{ac} and G_{grav} is made difficult by the analytical complexity of these formulas. It is possible, however, to deduce asymptotically approximate formulas for these integrals, the accuracy of which unlimitedly rises with t increase at certain values of dimensional variables. We shall proceed now to deduction of this asymptote. Naturally, this type of asymptote will be useful not the vicinity of wave front, but only in the internal region.

We start with G_{ac}. First of all we convert the integral in the right portion of (7.15). With this object we change the contour of integration; we shall draw it not as in Fig.7.1, but as in Fig.7.2. This could be done with the use of integrand function damping in the left half-plane. The advantages of this type of contour is that function e^{pt} gets exponentially damped in the left half-plane, and if we discard from the contour rectilinear "tails", leaving only as low as desired (but fixed) surroundings of special point with radius   , there will be an error exponentially damping with t increase, i.e., unreal in the main term asymptotes.

Let us deal first with the integral along the top loop. From what has been stated it follows that it is possible to expand the integrand function into series according to orders $p - i\tau_1$ and to take the least number of these series terms for obtaining the chief term of asymptotes. This least number is two with expansion into series of the exponent, since the first term being one does not have a branching point and the values of integrals on both the edges of the section will reciprocally disappear. Simply speaking, all functions not branching at point $p = i\tau_1$ could be substituted at this point by their values, and the exponential curve - by its exponent

$$-\frac{1}{8\pi^2 i} e^{pt} \frac{-C^{-1}\sqrt{ip + \tau_1}\sqrt{2\tau_1}}{-(\tau_1^2 - \tau_2^2)}.$$

The argument of radical $\sqrt{ip + \tau_1}$ should be taken at the top edge of the cut as $3\pi/4$, and at the lower edge - $-\pi/4$; $\sqrt{2\tau_1}$ is assumed to be positive. Now we bring in substitution of variable in the integral

$$p = i\tau_1 - x.$$

Then the sum of integrals on both the edges of the cut will be

$$\frac{1}{8\pi^2 i}\int_0^{\varepsilon} e^{i\tau_1 t} e^{-\alpha t} \frac{\sqrt{2\tau_1}}{C(\tau_1^2 - \tau_2^2)} (e^{i3\pi/4} - e^{-i\pi/4})\sqrt{\alpha}\, d\alpha.$$

The integral is taken here on a minor surrounding of zero $(0, \varepsilon)$, in which case there will be an error, exponentially low with t increase, if the integral is spread from 0 to ∞. Then the integral is easily calculated and found to be equal to

$$\frac{1}{4\pi^2 i} e^{i(\tau_1 t + 3\pi/4)} \frac{\sqrt{2\tau_1}}{c(\tau_1^2 - \tau_2^2} \frac{1}{t^{3/2}} \frac{\sqrt{x}}{2},$$

or

$$\frac{\sqrt{xgC}}{2\pi^{3/2}(x-2)^2 g^2} t^{-3/2} \exp\left[i\left(\frac{xg}{2C} t + \frac{\pi}{4}\right)\right].$$

The integral on lower loop is, apparently, complexly congugated with this one. Therefore,

$$G_{ak} = \frac{1}{(\pi gt)^{3/2}} \frac{\sqrt{xC}}{(x-2)^2} \cos\left(\frac{xg}{2C} t + \frac{\pi}{4}\right). \quad (7.18)$$

The main part of the integral is found. It dampens as $t^{-3/2}$. The next terms dampen even quicker. Primarily the attention is drawn to the fact, that the main part is independent of any dimensional coordinate. Thus, after some time from the acoustical part of the resolution remains some quickly damping background, varying cophasaly and with similar amplitude in big volume. In fact, the damping is even quicker, if we take into account the wave reflected from the earth's surface, i.e., the uneven continuation of the starting conditions. The effect of operator G on odd function could also be written as:

$$Gu(x, y, z) = \int_0^\infty \int\int_{-\infty}^\infty \left[G(x - x_1, y - y_1, z - z_1, t) - \right.$$

$$\left. - G(x - x_1, y - y_1, z + z_1, t)\right] dx_1 dy_1 dz_1.$$

Since the main asymptotic term G_{ac} does not depend on z, then in this formula the incident and reflected waves are reciprocally destroyed. Thus, we can only say, that starting from the moment, when both for incident and reflected waves it is possible to apply asymptotes, the acoustical part of the solution dampens quicker than $t^{-3/2}$. We did not investigate the next terms of asymptotes.

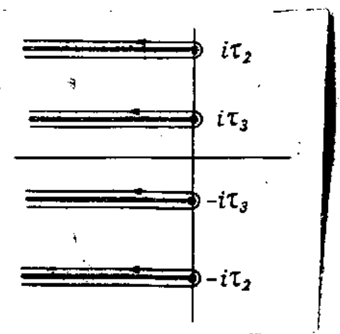

Fig.7.3 - Integration contour for asymptotic evaluation of G_{grav}.

Passing to investigation of G_{grav}. Here we shall carry out the same contour deformation, as in the preceding case (Figure 7.3). First we take the integral on the loop, by-passing point $i\tau_3$ (and the complex conjugate, on the loop by-passing point $-i\tau_3$). The reasoning will be the same as in the preceding case. Here it will be even simpler: in the expansion of exponent it is possible to take the first term - one, as the branching remains in denominator. The integrand function becomes:

$$-\frac{}{8\pi^2 i} e^{pt} \frac{1}{\sqrt{\tau_2^2 - \tau_3^2}\sqrt{x^2 + y^2 + z^2}\,\sqrt{2\tau_3(ip + \tau_3)}}$$

The argument of radical $\sqrt{ip + \tau}$ is equal to $-\pi/4$ at the lower edge of section and $3\pi/4$ - on the upper edge. We bring-in the same substitution of variables, as before

$$p = i\tau_3 - x.$$

Then the integral becomes

$$- \frac{1}{8\pi^2 i} \int_0^{\varepsilon} e^{i\tau_3 t}\, e^{-\alpha t} \frac{e^{i\pi/4} - e^{-i3\pi/4}}{(\tau_2^2 - \tau_2^2)\,(x^2 + y^2 + z^2)\; 2\tau_3 \alpha}\, d\alpha.$$

Calculating this integral and adding it up with the complexly conjugated, we get

$$G_{rpab}^{(1)} = \frac{1}{(8\pi^3 t)^{1/2}} \frac{\cos(\tau_3 t + \frac{3\pi}{4})}{\sqrt{(x^2 + y^2)\,(\tau_2^2 - 1^2)\;\tau_3}}. \tag{7.19}$$

Now let us change-over to the second part of G_{grav}, to the integral on loop around the point $\pm\tau_2$. For asymptotic calculation of this integral the p in integrand function could be substituted by its value at point of branching $i\tau_2$ everywhere, except the exponent e^{pt}, which provides for the damping of integrand function and the binomial $p^2+\tau_2^2$. The latter, by the way, could be simplified, by assuming $(\tau_2 + ip)(\tau_2 - ip) \approx 2\tau_2(\tau_2 + ip)$. As a result the integral becomes

$$- \frac{1}{8\pi^2 i} \int e^{pt} \frac{\exp\left[-c^{-1}\sqrt{x^2 + y^2 + z^2}\, i \sqrt{\frac{(\tau_1^2 - \tau_2^2)\,(\tau_2^2 - \tau_3^2)}{2\tau_2\,(ip + \tau_2)}}\right]}{i\sqrt{2\tau_2\,(ip + \tau_2)}\,\left(\tau_2^2 - \tau_3^2\right)\sqrt{x^2 + Y^2 + z^2}}\, dp,$$

or, if we introduce a new variable of integration $p = i(\tau_2 + \alpha)$,

$$\frac{e^{i\tau_2 t}}{b} \int_T e^{i\alpha t} \frac{e^{-\frac{a}{\sqrt{\alpha}}}}{\sqrt{\alpha}}\, d\alpha \tag{7.20}$$

where,

$$a = C^{-1}\sqrt{x^2 + y^2 + z^2}\ \frac{\sqrt{(\tau_1^2 - \tau_2^2)(\tau_2^2 - \tau_3^2)}}{2\tau_2},$$

$$b = 8\pi^2\sqrt{2\tau_2(\tau_2^2 - \tau_3^2)}\ \sqrt{x^2 + y^2 + z^2} \qquad (7.21)$$

The integration contour on plane α is shown in Figure 7.4. For the correct selection of branches it is necessary to assume $\sqrt{\alpha}$ as positive for positive α.

The integral obtained unfortunately, cannot be evaluated by such simple means as before, i.e., expansion of integrand function into series in surroundings of the special point, due to the fact, that now connected with the branching point is a real feature. It will be necessary to use the method of crossing. Having written the integrand

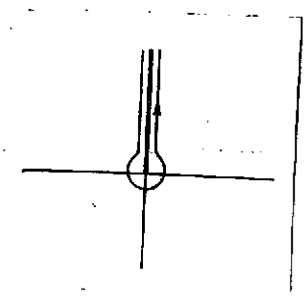

Fig.7.4 - Part of contour, shown in Figure 7.3, on plane .

function as

$$e^{H(t,\alpha)},\ H(t,\alpha) = i\alpha t - \frac{a}{\sqrt{\alpha}} - \frac{1}{2}\ln x,$$

It will be necessary to draw the contour of integration through the cross-over point, i.e., through a point, at which the real part of exponent would be greater than throughout the rest of the contour, and in such a direction, that the imaginary part of the exponential curve on this contour in surroundings of the crossover point would have been constant. We shall investigate first the real part of the exponent. Assuming $\alpha = re^{i}$, argument may vary in accordance with the section drawn in plane α from $-3\pi/2$ to $\pi/2$. Then

$$\mathrm{R\,e}\,H\,(t, \alpha) = -\,rt\,\sin\varphi - \frac{a}{\sqrt{r}}\cos\frac{\varphi}{2} =$$

$$= -\cos\frac{\varphi}{2}\left(2rt\,\sin\frac{\varphi}{2} + \frac{\alpha}{\sqrt{r}}\right)$$

(The logarithmic term need not be taken in the estimate, as it is asymptotically negligible in comparison with preceding term at low α). This formula is equal to zero, if $\cos\frac{\varphi}{2} = 0$, i.e., on the negative semiaxis, $\varphi = -\pi$, and on the curve $2rt\,\sin\frac{\varphi}{2} + \frac{a}{\sqrt{r}} = 0$. These two curves are shown in figure 7.5. They break up the plane into two zones: in one the real part of exponent is positive, in the other - negative. The first zone on the sketch is hatchured. Intersection of the two curves occurs at point

$$\alpha_1 = -\left(\frac{a}{2t}\right)^{2/3}.$$

We draw the contour of integration as shown in the figure. Throughout the contour Re $H < 0$, and at point α_1 Re $H = 0$, i.e., at this point the Re H is highest. We shall show that it is possible to

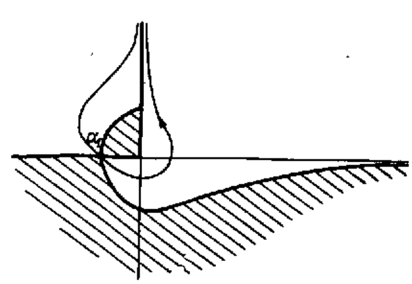

Fig.7.5 - Contour deformation for the application of crossover method.

draw the contour in such a direction that ImH in surroundings of α_1 will be constant, i.e., that this point is the sought for crossover point. In surroundings of point α_1 we expand $H(t,\alpha)$ into series according to orders of $\alpha - \alpha_1$; $H(t, \alpha_1) = 3i\ (a^2t/4)^{1/3} - \ln[-i(a/2t)^{1/3}]$. The next term of the first order relatively to $(\alpha - \alpha_1)$ converts into zero, since $\partial H(t, \alpha_1)/\partial\alpha = 0$. The square term is equal to $\frac{(\alpha - \alpha_1)^2}{2}$ $\left(-\frac{3a}{4\alpha_1^{5/2}} - \frac{1}{2\alpha_1^2}\right)$. The second term in brackets could be asymptotically disregarded. Now we take the direction of contour in such a way, that the square terms along the contour will be real. For this there should be

$$(\alpha - \alpha_1)^2 = \frac{\alpha - \alpha_1{}^2\ \alpha_1^{5/2}}{\alpha_1^{5/2}}$$

or $\alpha - \alpha_1 = \pm|\alpha - \alpha|_1/e^{-i\pi/4}$, i.e., contour shown in figure 7.5 should intersect the real axis at point α_1 at an angle $-\pi/4$.

Let us return to integral (7.20). For asymptotic evaluation of its behavior the significance is only of the surroundings of the cross-over point α_1, since the remaining portion exponentially decreases. Whereas in the surrounding of this point it is possible to write, that the integral is asymptotically equal to

$$e^{-3i\ (a^2t/4)^{1/3} + i\pi/4}\left(\frac{a}{2t}\right)^{-1/3}\int e^{-\ (3a/8)\ (a/2t)^{-5/3}\ \sigma^2}\, d\sigma\ .$$

Here we have taken as a varibale integration $\sigma = \alpha - \alpha_1)\ e^{i\pi/4}$. The limits of integration without any adverse effect could be taken $-\infty$ and ∞, changing integral to exponentially decreasing summand. The integral obtained is easily calculated; it is equal to

$$\sqrt{\frac{4\pi}{3t}}\ e^{-i\left(3\sqrt[3]{\frac{a^2t}{4}}-\frac{\pi}{4}\right)}.$$

It only remains to remember the multiplier, which was in front of the integral in formula (7.20) and to add to the obtained term the complex adjoint (integral on symmetrical loop around $-i\tau_2$). We have, finally

$$G^{(2)}_{rpab} = \sqrt{\frac{\pi}{3t}}\ \frac{1}{b}\cos\left(\tau_2 t - 3\left(\frac{a^2t}{4}\right)^{1/3} + \frac{\pi}{4}\right). \qquad (7.22)$$

Thus, the asymptotes of the gravitational part in the function of effect is shown by the formula

$$G_{rpab} = G^{(1)}_{rpab} + G^{(2)}_{rpab}. \qquad (7.23)$$

The gravitational part of the effective function dampens as $t^{-1/2}$, i.e., is considerably slower, than the acoustical. It is considerably non-uniform in coordinates and is anisotropic. This asymptote decays in the approach to vertical direction, where it absolutely loses any meaning.

But what is the behavior of G_{grav} vertically, at $x^2 + y^2 = 0$? It is easy to see, that in these conditions $\tau_2 = \tau_3$ and the calculation of G_{grav} is confined to integration along the contour, passing around the pole, i.e., to calculation of substraction at point $p = i\tau_2 = i\tau_3$

$$G_{rpab} = -\frac{1}{8\pi^2 i}\oint e^{pt}\ \frac{\exp\left[-C^{-1}z\sqrt{\tau_1^2 + p^2}\right]}{(\tau_2^2 + p^2)\,z}\,dp + \text{complex conjugated}$$

term.

This produces

$$G_{rpab} = - \frac{1}{4\pi z} \sqrt{\frac{xH}{(x-1)\,g}} \exp\left[- \frac{|x-2|}{2xH} z \right] \sin \sqrt{\frac{(x-1)g}{xH}}\, t. \tag{7.24}$$

This is not asymptote, but exact solution. We can see, that on vertical line there is always undamped standing wave[1].

(5) NUMERICAL ESTIMATE RESULTS OF THE WEIGHTING FUNCTION:

Asymptotes, obtained in the preceding paragraph, were useful with high t values, i.e., considerable time after passage through a viven space point of the wave front. It is of interest to estimate the weighting function throughout the whole zone, specially in the vicinity of the front. This will make it possible, besides everything else, to evaluate the zone, where the asymptoes give satisfactory results. We give the results of such estimates.

Figure 7.6 - 7.13 show contour lines of acoustical and gravitational parts of average function for different moments of time. The initial disturbance was taken at 30 km height, and the following functions were plotted.

$$G\,(x - x_1,\ y - y_1,\ z - z_1,\ t) - G\,(x - x_1,\ y - y_1,\ z + z_1,\ t)$$

for G_{ac} and G_{grav}. Fig. 7.6 shows contour lines of G_{ac} at t = 1 min, Fig.7.7 - these of G_{grav}. The disturbance is propagating at velocity c = 18 km/min., therefore, during 1 min it did not have time to reach the earth's surface. The contour lines are similar to concentric

1. In Ross' article (1961) a similar problem is being resolved, but for alayer of compressible fluid with open surface. The equations there are somewhat simpler of second time order. Asymptotic investigation leads to similar results: immediately behind the wave front the main significance is of elasticity, whereas in the rear the oscillations are similar to those, which would have been in incompressible fluid.

surroundings. However, even with low t value the difference is clear in the behavior of acoustical and gravitational parts. While the G_{ac} contour lines concentrate at the periphery and the field in the internal part varies continuously, the gravitational part of the mean function G_{grav} behaves in an entirely different way. The zone of sharp variations remains near the point of initial disturbance, and farther on the function gradually drops. This difference is specially clearly defined in vertical sections (shown on the right), drawn at a distance of 2 km from the point of initial disturbance.

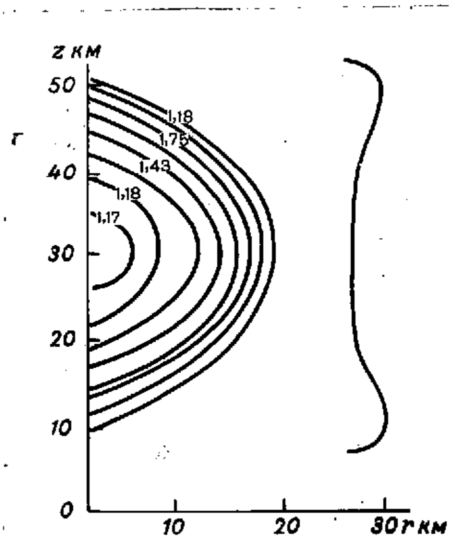

Fig. 7.6 - G_{ac} contour lines at t = imin. $r = \sqrt{x^2 + y^2}$

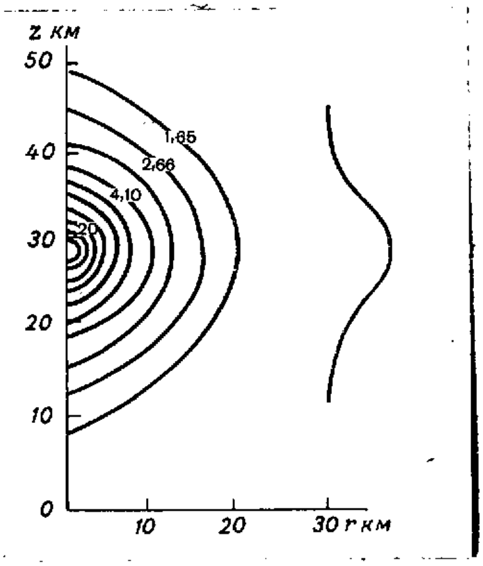

Fig.7.7 - G_{rav} contour lines at t = 1 min.

The next four figures pertain to the time t = 5 min. Fig. 7.8 pertains to acoustical part. The sphericity of level surface is disturbed by the reflection of the earth's surface, but non-the-less, remains noticeable. Figure 7.9 shows vertical section of this mean function, drawn through the point of initial disturbance, and the reflection was not estimated here. It is possible to see the internal zone of G_{ac} invariability and quick drop toward the edge. Figure 7.10 shows gravitational part. This function in no way resembles the acoustical. The change of phase is specially typical. Figure 7.11 shows for comparison weighting function, claculated from asymptotic formulas, obtained in the preceding paragraph. The qualitative convergence could be taken as successful.

Figure 7.12 pertains to the gravitational part of the mean function G_{grav} at t = 15 min, and figure 7.13 - corresponding asymptotes. With the passage of time the number of nodal surfaces increases and the pattern becomes multipetal. Both the figures coincide satisfactorily. The number of lobes is the same. Moreover, the coincidence takes place not only in the center, but for on the periphery. All this goes to show, that the asymptotic formulas could be used for mean function in a wide range of conditions.

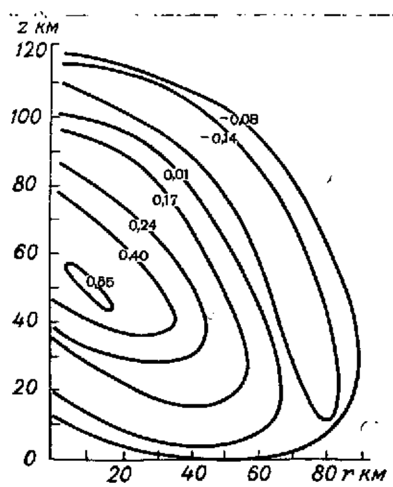

Fig.7.8 - G_{ac} contour lines at t = 5 min.

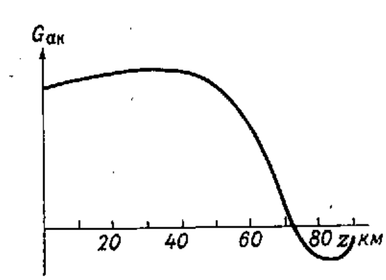

Fig.7.9 - Vertical G_{ac} section; r = 0.

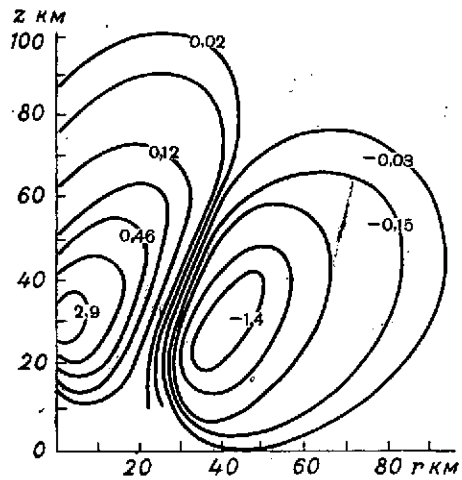

Fig.7.10 - G_{grav} contour lines at t = 5 min.

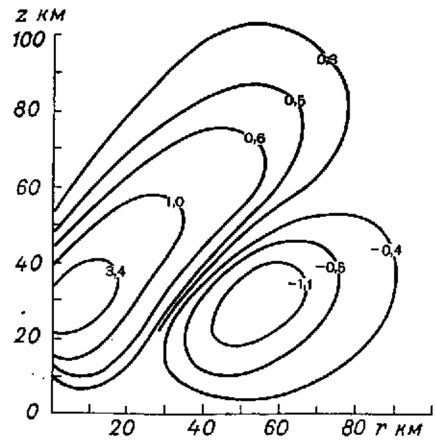

Fig.7.11. Contour lines of G_{grav}, calculated from asymptotic formula at t = 5 min.

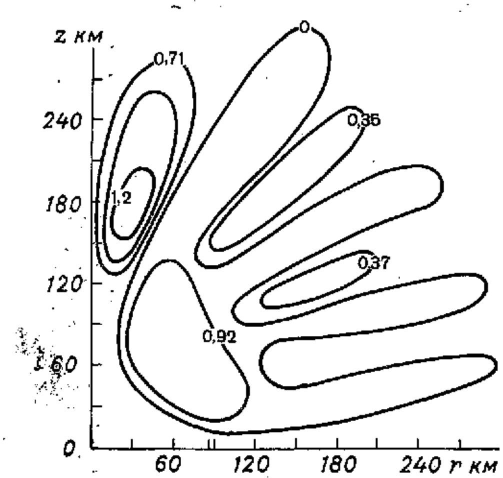

Fig.7.12 - G_{grav} contour lines at t = 15 min.

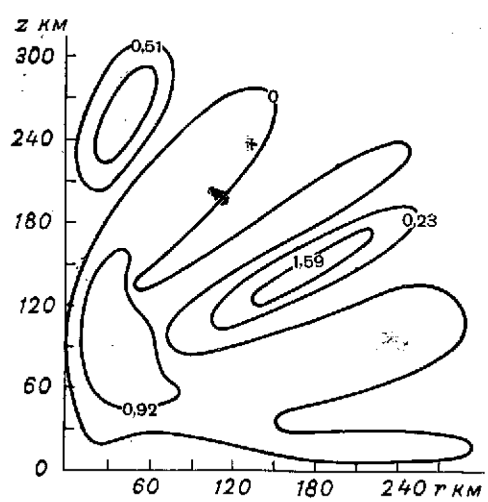

Fig. 7.13 - Asymptotic expression for G_{grav} at T = 15 min.

6. FORMULATION OF PROBLEM ON PROPAGATION OF DISTURBANCES IN THE CASE OF ACTUAL TEMPERATURE PROFILE AND SPHERICAL EARTH:

The problem, about which we shall speak in this paragraph, is practically the most important out of all the applications of developing theorem. There is a good amount available of the literature, which throws some light or touches upon this question. Until recently the only manifestations of the global disturbance propagation from a concentrated source, fixed by observers and to a certain extent investigated qualitatively, were the two magnificient natural explosions - the explosion of Krakatoa volcano in 1883 and the explosion of the huge Tunguss meteorite in 1908. Pressure surges were marked by barographs throughout the terrestrial globe, and the waves passed around the earth several times. It was possible to measure the propagation velocity of the wave front. It was found to be about 317 m/sec, which corresponds to h = 10 km, obtained from formula $v = \gamma\ gh$.

Later on these data served as empirical basis for working out theoretical models of atmosphere. (We have seen, that this h value conforms very well also with our model, based on standard CIRA atmosphere 1961). At that stage the more well known theoretical work, which threw some light on the mechanism of disturbance propagation from the point source, was of Peckeris (1939, 1948), and also of Scorer (1950) and of Jacchia and Kopal (1952). These models are still very sketchy.

During the subsequent period as a result of hydrogen bomb explosions the accumulation, unfortunately, was of too vast observation material. The interest for investigating this material empirically and theoretically was shown primarily by Japanese investigators (see for instance Obs.division, 1955), specially Yamamoto (1957). A note should be taken also of Wexler and Hass article (1962) with multiple, very graphic figures of wave front, propagating from the point of hydrogen bomb explosion, and the recently published work of Wickersham (1966).

Many theoretical works have appeared. In majority of them the atmosphere is presented as composed of a high number (upto 20-40) isothermal layers. Here a note should be made regarding the work of Hunt, Palmer and Penney (1960), where the number of layers is still not high, and the subsequent work of Press and Harkrider (1962) and Pfeffer and Zarichny (1963), also the works of Pierse (1963, 1965) and van Hulsteyn (1965) From our point of view, the most mathematically accurate formulation of the problem is of Weston (1961, 1962), although in other works also the physical results obtained are highly interesting. These results have been partially mentioned in preceding chapters.

We shall construct now our own version of the theory of disturbance propagation from the instanteneous point source, based on our previous analyses. We shall only write the formulas. Calculations from them were not carried out.

We shall take as basis formulas of preceding chapter. At the starting moment, as has been stated, starting moment, as has been stated, function y $\left(= X \sqrt{\bar{P}/\bar{P}_o}\right)$ should be preset and its three derivatives with respect to t i.e., conditions (6-4). In this case the functions $y^{(k)}$ are not at all arbitrary. The arbitrary are the starting fields u, v, w, p, ρ . They express the divergence X quite obviously, X_t from formula (6-2) and the two next derivatives from the general formula (6-3). This formula combines the two. One of them is obtained by substituting instead of $\tilde{f}_1$, $\tilde{f}_2$, φ functions X, X_{tt} φ , which we denote by $\tilde{f}_1^{(c)}$, $\tilde{f}_1^{(c)}$, φ (c). The second formula is obtained by substituting $\tilde{f}_1 = X_t$, $\tilde{f}_2 = X_{ttt}$, $\varphi = -\frac{1}{\rho}(P_z + g\rho)$. These values we denote by $\tilde{f}_1^{(d)}$, $\tilde{f}_2^{(d)}$, $\varphi^{(d)}$. Since y is distinct from X by normalizing multiplier $\sqrt{\bar{P}/\bar{P}_o}$, the starting values will be:

$$y^{(o)} = f_1^{(c)},\ y^{(1)} = f_1^{(d)},\ y^{(2)} = f_2^{(c)},\ y^{(3)} = f_2^{(d)}\ ,$$

where

$$f_{1,2} = \tilde{f}_{1,2} \sqrt{\frac{\bar{P}}{P_o}} \tag{7-25}$$

The starting values are expanded into spherical functions, after which, as we know, for each component (n, s) it is necessary to find expansion by natural oscillations:

$$y^{(o)}_{n,s} = \sum_j C_{n,s,j} Y_{n,j}(X)$$

$$Y^{(2)}_{n,s} = \sum_j \left(-\sigma^2_{n,j}\right) C_{n,s,j} Y_{n,j}(X)$$

and

$$y^{(1)}_{n,s} = \sum_j d_{n,s,j} Y_{n,j}(X),$$

$$Y^{(3)}_{n,s} = \sum_j \left(-\sigma^2_{n,j}\right) d_{n,s,j} Y_{n,j}(X).$$

Finally the resolution will be

$$y = \sum_{n,s,j} e^{is\varphi} P_n^{|s|}(\cos\theta) Y_{n,j}(X) \left(a_{n,s,j} e^{i\sigma_{n,j} t} + b_{n,s,j} e^{-i\sigma_{n,j} t} \right) \qquad \ldots.(7.25)$$

where

$$a_{n,s,j} = \frac{1}{2}\left(C_{n,s,j} + \frac{d_{n,s,j}}{i\,\sigma_{n,j}} \right), \quad b_{n,s,j} = \frac{1}{2}\left(C_{n,s,j} - \frac{d_{n,s,j}}{i\,\sigma_{n,j}} \right)$$

Formulas for factors c, d were given in chapter 6. If we substitute there the term $f_1^{(c,} \, _2^{d)}$, we shall have after some conversions[1]

$$C_j,\ d_j = x\bar{P}_o \int_0^{\infty} \sqrt{\frac{\bar{P}}{P_o}} \quad X$$

$$\left\{ g \left[Y_j' + \left(\frac{1}{h_j} - \frac{1}{2H} \right) y_j \right] \left(\frac{1 + H'}{H} \varphi^{(c,d)} \cdot f_1^{(c,d)} \right) \right. +$$

$$X \; \frac{ + \left(\sigma_j^2 - \frac{g}{h_j} \right) \left(\frac{1}{\chi H} \varphi^{(c,d)} - f_1^{(c,d)} \right) \left. Y_j \right\} dz }{ \xi_j \left(\frac{g}{h_j} - \sigma_j^2 \right) \xi_j } \qquad (7\text{-}26)$$

---

1 The conversions mean, that into formulas (6-50) instead of $f_{2(C,d)}^{(c,d)}$ it is necessary to substitute their terms (6-3) by $f_1^{(c,d)}$ and . After this integration is done by parts so that under the sign of integral there will not be derivative functions f_1 and , but only these functions. In integration by parts appear marginal terms and it is not obvious at once that they are zero. In fact, they are zero, because the starting data at z = 0, should meet the following conditions: there should be conversion into zero of w and also of w time derivatives, which could be expressed by means of a system of equations by the starting fields u,v,w,p, . Thus, $w_1 = -\frac{1}{\rho}(pz+g\rho)$ and $w_{tt} = -/(\chi - 1)gX - c^2X_z + gw_z)/$ should convert into zero. This is what assures disappearance of marginal terms in formula (7-26).

Incidentally it is precisely this conversion we had in view at the end of preceding chapter, when speaking about factors c, d converting into zeros for Peckeris solutions. It becomes immediately clear from formula (7.26).

Let us now take a particular case of setting starting conditions, when only density disturbances are distinct from zero. Then $f^{(c)}_{1,2} = 0$, and therefore, all the factors $cj = 0$; $\varphi^{(d)} = -g\rho/\ddot{\rho}$, $\varphi^{(d)}_{z} = f^{(d)}_{1}$. Substituting these values in (7.26) and integrating by parts, so that this formula does not contain derivatives $\varphi^{(d)}_{z}$, but only $\varphi^{(d)}$, we will have

$$d_j = \frac{x\bar{p}o \int\limits_{o}^{\infty} \sqrt{\frac{\bar{p}}{\bar{p}o}} \left[y'_j + \left(\frac{1}{h_j} - \frac{1}{2H} \right) y_j \right] \varphi \, dz + x\bar{p}o\sigma_j^{-2} \left(\sigma_j^2 - \frac{g}{h_j} \right) \varphi(0) \; y_j(0)}{\left(\frac{g}{h_j} - \sigma_j^2 \right) \mathcal{E}_j} \qquad (7.27)$$

Hence, we shall assume that the close to surface disturbance values, i.e., $\varphi^{(0)}$, are zero. Now we make one more simplifying assumption, that the starting conditions are of the type δ = function:

$$\varphi = K\,\delta(z - z_*) \sum_{n=o}^{\infty} \left(n + \frac{1}{2} \right) P_n (\cos \theta). \quad (7.28)$$

then,

$$d_j = \frac{x\sqrt{\bar{p}o\bar{p}_*} \left[y'_j + \left(\frac{1}{h_j} - \frac{1}{2H} \right) y_j \right]_* \left(n + \frac{1}{2} \right) K}{\left(\frac{g}{h_j} - \sigma_j^2 \right) \mathcal{E}_j}$$

(7.29)

The asterisk signifies, that the corresponding values is taken at the height of explosion z_*.

In principle, the solution has been obtained. But physically the interest is not of function y, which is not directly evident, but pressure. Using the y and pressure relation (see beginning of chapter 5), it is easy to obtain,

$$p = x^2 \bar{p}oK \sqrt{\bar{p}_* \bar{p}} \sum_{n,j} \left(n + \frac{1}{2}\right) P_n (\cos \theta) \cos \sigma_{n}, t \times$$

$$\frac{\left[y'_{n,j} + \left(\frac{1}{h_{n,j}} - \frac{1}{2H}\right) y_{n,j}\right]_* \left[{}^{2}_{n,j} y_{n,j} + g\left(y'_{n,j} - \frac{1}{2H} y_{n,j}\right)\right]}{\sigma^2_{n,j} \left(\sigma^2_{n,j} - \frac{g}{h_{n,j}}\right)^2_{6} \quad j} \qquad (7.30)$$

This resolution includes constant K, which has to be selected from some normalizing conditions. One of the possible methods most frequently applied (for instance, by Weston), is to preset amplitude of disturbance at some single point, for instance, directly above the point of explosion. These data could be taken from observations. The amplitude here is taken somewhat conventionally, since the resolution is not sinusoidal. Immediately after the passing of front through the observation point the pressure reaches maximum then follow several gradually damping waves. In formulas (7.30) and (7.31) the summation is double. The following order of summation is the most conveninent. First the summation is done by n, along one mode of vertical equation, and then by all (or practically, several) moede, analysing series (7.30) and (7.31), it can be seen, that the inclusion of very low and very high wave numbers n is negligible, so that it is possible to leave only certain range of mean n values.

If the low n numbers are discarded, the Legendre polynomials in formula (7.30) could be substituted by their asymptotic values at high numbers

$$P_n (\cos \theta) - \frac{\cos \left(n + \frac{1}{2}\right) \theta - \frac{\pi}{4}\Big]}{\sqrt{2\pi n \sin \theta}} \tag{7.31}$$

This presentation will be disturbed only in the direct vicinity of poles, mole exactly, it holds true at $\varepsilon/n < \theta < \pi - \varepsilon/n$, where is random fixed constant. The lower we take this constant, the rougher will be the residual term. The sum (7.30) we shall write as

$$p = \sum_n \cos \left[\left(n + \frac{1}{2}\right) \theta \frac{\pi}{4} - \sigma_{nj} t \right] f(n) + $$
$$+ \sum_n \cos \left[\left(n + \frac{1}{2}\right) \theta \frac{\pi}{4} + \sigma_{n,j} t \right] f(n). \tag{7.32}$$

Under the sign of the sum here is the product of fast oscillating function $\cos\left[\left(n + \frac{1}{2}\right)\theta - \frac{\pi}{4} \sigma t\right]$ or $\cos\left[\left(n + \frac{1}{2}\right)\theta - \frac{\pi}{4} + \sigma t\right]$ by function, changing very gradually. Since we are not taking the lowest n values, the spectrum of Laplace's equation will be very dense, almost continuous, between the two subsequent eigen values n the function changes very little.

Let us now take the sum along some characteristic curve. For any t moment and point θ the main significance in the sum is of several summands corresponding to wave packet with group velocity aθ/t, i.e., passing the given point at a given moment, and also to wave packets passing through given point not directly from the source, but after passing around the earth several times. This could be proved by the method of stationary phase, applied, for instance, by Weston (1961). The principle of stationary phase was invented by Kelvin specially for similar situations, he also has introduced the concept of group velocity by means of this principle. In contrast to the usual application, we apply now the principle of stationary phase to the sum, without substituting it previously by integral, which would have been baseless due to the presence of fast-oscillating function, the period of which is quite commensurable with the spacing of summation.

Let us take the first of the summands in (7.32). Taking any n value, we assume it is , and analyse several terms of the series with values n close to . We use expansion in accordance with Taylor's formula

$$\cos\left[\left(n+\frac{1}{2}\right)\theta-\frac{\pi}{4}-\sigma_n t\right]\simeq \mathrm{Re}\left\{e^{i\left[\left(\nu+\frac{1}{2}\right)\theta-\frac{\pi}{4}-\sigma_\nu t\right]}\times e^{i(n-\nu)\left[\theta-\frac{\partial\sigma_\nu}{\partial n}t\right]}\,e^{-i\frac{(n-\nu)^2}{2}\frac{\partial^2\sigma_\nu}{\partial n^2}t}\right\}. \tag{7.33}$$

Since the $f^{(n)}$ varies very gradually, we substitute the $f^{(n)}$ values by $f(\nu)$. The values is stationary, if

$$\exp\left\{i\,(n-\nu)\left[\theta-\frac{\partial\sigma_\nu}{\partial n}t\right]\right\}=1,\ \text{т.е.}$$

$$\theta-t\,\frac{\partial\sigma_\nu}{\partial n}=2\pi\alpha\,,\ \alpha=0,\ 1,\ \ldots$$

In this case the formula for wave packet, i.e., for the sum in n values, similar to , could be converted as follows:

$$\sum \cos \left[\left(n + \frac{1}{2}\right) \theta - \frac{\pi}{4} - \sigma_{n,j} t \right] f(n) \approx$$

$$\approx \mathrm{Re} \left\{ e^{i\left[\left(\gamma + \frac{1}{2}\right) \theta - \frac{\pi}{4} - \sigma_{\gamma,j} t \right]} \sum e^{-\frac{i (n - \gamma)^2}{2} \sigma''_{\gamma,j} t} \right\} f(\gamma) \approx$$

$$\approx \mathrm{Re} \left\{ e^{i\left[\left(\gamma + \frac{1}{2}\right) \theta - \frac{\pi}{4} \quad \sigma_{\gamma,j} t \right]} \times \right.$$

$$\left. \times \int e^{-\frac{i (n - \gamma)^2}{2} \sigma''_{\gamma,j} t} \, dn \right\} f(\gamma) =$$

$$= \sqrt{\frac{2\pi}{|\sigma''_{\gamma,j}| t}} \cos \left[\left(\gamma + \frac{1}{2}\right) \theta - \frac{\pi}{4} + \quad \sigma_{\gamma,j} t + \right.$$

$$\left. + \frac{\pi}{4} \operatorname{sgn} \sigma''_{\gamma,j} \right] f(\gamma). \tag{7.34}$$

For the second summand in (7.33) the stationary points are those, where $\theta + t \partial \sigma_{\gamma} / \partial n = 2\pi\alpha$, $\alpha = 0, 1, 2, \ldots$ and the corresponding wave packets contribute

$$\sqrt{\frac{2\pi}{|\sigma''_{n,j}| t}} \cos \left[\left(\gamma + \frac{1}{2}\right) \theta - \frac{\pi}{4} + \sigma_{\gamma,j} t + \frac{\pi}{4} \operatorname{sgn} \sigma''_{\gamma,j} \right] f(\gamma). \tag{7.35}$$

Considering, that number n is bound with wave number k by relation k n/a, we may finally formulate the obtained result in the following way. At present θ and t at each mode of vertical equation we have to find points, where group velocity is

$$\frac{\partial \sigma}{\partial k} = \frac{a\theta}{t} + 2\pi\alpha, \ \alpha = 0, 1, \ldots \tag{7.36}$$

or

$$\frac{\partial \sigma}{\partial k} = - \frac{a\theta}{t} + 2\pi\alpha, \quad \alpha = 0,\ 1,\ \ldots \qquad (7.37)$$

amd each of these points contributes into formula for pressure (7.34) or (7.35), where

$$f(\gamma) = \frac{x^2 \bar{p}o \sqrt{\bar{p}_* \bar{p}\, K} \quad \sqrt{\gamma}}{2\sqrt{2\pi}} \times$$

$$\times \frac{\left[y'\gamma_{,j} + \left(\frac{1}{h\gamma_{,j}} - \frac{1}{2H}\right) y\gamma_{,j}\right]\left[\overset{2}{\sigma}\gamma_{,j} y\gamma_{,j} + g\left(y'\gamma_{,j} - \frac{1}{2H} y\gamma_{,j}\right)\right]}{\left(\sigma\gamma^2_{,j} - \frac{g}{h\gamma_{,j}}\right)^2 \sigma\gamma^2_{,j}\, \zeta\gamma_{,j}}. \qquad (7.38)$$

The physical meaning of (7.36), (7.37) is quite clear - group velocity is such, that the corresponding wave packet during time t reaches point θ, directly from the shot point, or having passed around the earth once or several times. The formulas obtained should be summed up on all the modes, where there are points with this group velocity. As the formulas show, the essential point here is, that $\sigma'' \neq 0$, i.e., the variability of group velocity or dispersion. Hence it is at once clear, that the reasons given are suitable only for those θ, t, for which a θ/t is lower than the maximum group velocity 315-317 m/sec.

But if the investigation has to be of the head of the wave, i.e., the surroundings of its propagation front, the stationary phase method is not applicable any more. Here we have to calculate the sum total throughout the main non-dispersing complex mode $h \approx 10$ km. It shoudl be mentioned, that this sum could be of significance also for the area located quite far in the rear of the front. This will happen, if the source, i.e., the shot is not high above the earth's surface. This is indicated by the multiplier $\left[y' + \left(\frac{1}{h} - \frac{1}{2H}\right) y\right]$. As we know, the basic mode h = 10 km very quickly dampens with height for low frequencies (periods over 10 min), and for high frequencies concentrates at height 17 km and just as quickly dampens at high altitudes. With high altitude explosions the basic mode cannot be excited at high rate. On the contrary, in this case there is excitation of higher modes, both acoustical and gravitational.

We have given formulas of solution with one version of preseting the starting conditions - when the starting density is preset in the form of - function, and the remaining quantities at the starting moment are equal to zero. Of course, the general formulas, given earlier, make it possible to resolve any starting problem. Now just a few words in defence of the selected starting conditions. At the instant of the shot the temperature in the source instantly rises. The density with this varies insignificantly, therefore, in proportion to temperature the pressure rises also. The shock wave withdraws from the shot point and beyond it the pressure levels out. The temperature remains high at low pressure, i.e., the density drops at high rate. This is the moment, that we are taking as starting point for the solution of our problem, since the shock wave is not included in our analysis, being an essentially non-linear and small scale event. As the starting condition it is also possible to take, as it is done frequently, the presetting of velocity divergence as a -function. Generally, the selection of correct starting conditions is a matter of explosion physics, the same as determination of the share of the explosion energy, which goes on formation of our large-scale waves, and not converting, for instance, into energy of shock wave, dispersing in the vicinity of the source.

CHAPTER - 8

ACCOUNTING FOR THE MEDIUM WIND IN THE PROBLEM OF FREQUENCIES ON THE ATMOSPHERE'S FREE OSCILLATIONS.

Throughout the theory, developed in preceding chapters, equations of dynamics were linearised in relation to a certain medium state, which was that of calm, i.e., there was no presence of medium wind. At the same time it is known, that the wind affects the propagation of waves in the atmosphere. The second important sphere of events, where the effect of medium wind could have been felt, is the theory of tides in the atmosphere. We have mentioned, that one of the possibilities to explain in which way the solar tides in the atmosphere happen to be of higher intensity than the lunar, whereas the tide-forming power of the sun is weaker than that of the moon, is the theory of resonance.

Since the difference between the frequencies of solar and lunar tides is very negligible, about 3%, this theory requires very precise "tuning" of natural frequency to frequency of solar tides, considerably more precise than these 3%. We know, that with our model of temperature stratification, namely, for the standard CIRA atmosphere 1961, the nearest eigen value is much too far from the one required by the resonance theory. The dimensionless frequency $f = \sigma/2$ is equal not to one (period of oscillations - semidiurnal), but to 0.96, i.e. 4% less. (If we compare not frequencies, but the values of equivalent dynamic depth h, then instead of the 7.9 km required by the theory we get 6.8 km). The question arises, whether the position is saved by the estimate of medium wind.

Generally speaking, the problems in which it is required to find the spectrum of the atmosphere's oscillations in the presence of certain medium, usually zonal wind, are very difficult and hardly touched upon problems, studied by the theory of dynamic stability. But in the theory of tides, as in some other important questions, for instance, in the theory of propagation of acoustico gravitational waves from the point source, it is possible, as we will see, to manage with considerably more modest means.

Assuming that an introduction is made into equation of a very low velocity of medium wind, each point of the spectrum experiences a small disturbance. The order of this disturbance will be such that the phase velocity of oscillation would have to change by a value, commensurable with the brought in wind velocity. With the increasing wind velocity the disturbances become more considerable and it may happen, that two eigen values will merge. With further increase of velocity they will separate in complex plane, i.e., the frequencies will become complex. There will be an appearance of instability. No matter how low the wind velocity is, there will be a zone in the spectrum, where the brought in disturbance will be sufficient for the merger of eigen value. This will occur in densification zone of the spectrum, for short gravitational waves. Here, as we know, phase velocities of characteristic solutions tend to zero.

The second zone, where the instability may appear immediately, is the zone of long and slow gravitational gyroscopic waves (internal Rossibi waves). As regards those zones of the spectrum, about which we were just speaking, primarily zones of semi-diurnal tides, here the spectrum points are sufficiently removed from each other. To estimate the wind effect here it is possible to use the perturbation theory.

We can mention several works, in which the perturbation theory is used with this object. Those are the works of Weston and van Hulsteyn (1962), and of Pierce (1965), who speak of quick waves, and Sawada (1966) in whose work these methods are applied in the theory of tides. However, in all these works the wind model is of the type, which does not prevent division of variables in equation. The aim of the present chapter is to show, in what way the perturbation theory may serve in the estimation of wind of any profile.

For the sake of simplicity we begin with the case of flat non-rotary earth. In the absence of moderate wind the natural oscillations satisfy the system of equations known to us (where zero index at the bottom means, that we are speaking of solution undisturbed by wind):

$$i\sigma_0 \bar{\rho} u_0 = -i\alpha P_0, \quad i\sigma_0 P_0 - g\bar{\rho}\omega_0 + \chi \bar{P} \chi_0 = 0,$$

$$i\sigma_0 \bar{\rho} v_0 = -i\beta P_0, \quad i\sigma_0 \eta_0 + \beta\bar{\rho}\omega_0 = 0. \qquad (8.1)$$

$$i\sigma_0 \bar{\rho}\omega_0 = -\frac{\partial P_0}{\partial z} - g\rho_0,$$

Here $\eta = P - C^2\rho$ - entropy. In this equation u_0, v_0, P_0, η_0 could be taken as real, ω_0 - purely imaginary, assuming now there is some field of moderate wind. For the sake of simplifying formula writing we assume (but this, generally speaking, is not essential), that the velocity vector is every where parallel to axis x, i.e., there is one component U, dependent on transversal coordinates y, z.

Let U = 0 at $|y| > Y$, i.e., the analysis is of stream. It may be assumed, that at U there is a certain minor parameter, by orders of which it is possible to expand the solutions. Then for the terms of the first order relatively to this parameter marked by index 1, we shall have a system:

$$i\sigma_0\bar{\rho}u_1 + i\sigma_1\bar{\rho}u_0 = -i\alpha P_1 - i\alpha\bar{\rho}U u_0 - \bar{\rho}U'_y v_0 - \bar{\rho}U_z\omega_0,$$

$$i\sigma_0\bar{\rho}v_1 + i\sigma_1\bar{\rho}v_0 = -\frac{\partial P_1}{\partial y} - i\alpha\bar{\rho}Uv_0,$$

$$i\sigma_0\bar{\rho}\omega_1 + i\sigma_1\bar{\rho}\omega_0 = -\frac{\partial P_1}{\partial z} - g\rho_1 - i\alpha\bar{\rho}U\omega_0$$

$$i\sigma_0 P_1 + i\sigma_1 P_0 - g\bar{\rho}\omega_1 + \varkappa\bar{P}\varkappa_1 \equiv -i\alpha U P_0,$$

$$i\sigma_0\eta_1 + i\sigma_1\eta_0 + \beta\bar{\rho}\omega_1 = -i\alpha U\eta_0. \qquad (8.2)$$

We multiply each equation of system (8.1) respectively by u_1^*, v_1^*, w_1^*, $P_1^*/\varkappa\bar{p}$, $g\eta_1^*/\varkappa\bar{p}\beta$ (where the asterisk means complex conjugation) and add up the obtained equations. We also multiply equations of system (8.2) by the same values and add up. We substitute the last sum by complexly conjugated and add the first sum. We integrate the obtained equation by z from 0 to ∞ and by y from -Y to Y, i.e., throughout the whole section of the stream. It is easy to check, that we will get

$$\sigma_1 E = -\alpha\int_{-Y}^{Y}\int_0^{\infty} U\, e\, dz\, dy - \frac{1}{2i}\int_{-Y}^{Y}\int_0^{\infty}\bar{\rho}U'_z\omega_0 u_0^* dz\, dy,$$

where e - density of energy, bound up with undisturbed solution

$$e = \frac{1}{2}\left[\bar{\rho}\ (|u_0|^2 + |v_0|^2 + |\omega_0|^2) + \frac{|p_0|^2}{x\bar{p}} + \frac{g|\eta_0|^2}{x\bar{p}\beta}\right.$$

and E - total energy, $E = 2Y\int_0^\infty edz$. In relation (8.3) only one quantity is marked by index 1, namely $_1$; the others pertain to undisturbed motion and are taken as known. From this formula we find disturbance of frequency $_1$. It is found to be real, as it should be within the applicability of the theory of disturbances. The right portion consists of two terms. The second, Reynolds number, could be taken as minior in those important cases, when the vertical velocity of natural oscillation is very low. In this case the formula simply confirms, that addition of $-\sigma_1/\alpha$ to phase velocity in the direction of flow is equal to flow velocity, averaged in the section, and the most interesting is the fact, that the weight in this averaging is the density of undisturbed oscillation energy.

Assuming also that $w_o = -i\tilde{w}_o$, so that all the variables u_o, v_o, $\tilde{w}_o$, p_o, η_o could be taken as real. Then the Reynold's number will be $\frac{1}{2}\iint \bar{\rho}\, U'_z\, w_o u_o dzdy$.

Exactly the same can be done in the case of the model of spherical rotatory earth. Assuming u is zonal component of velocity, v - meridional, $1 = 2\omega \cos \Theta$ (where Θ is polar angle), relationship lengitude φ exponential, i.e., has the nature of exp (is φ), $iv_0 = \tilde{v}_o$, $iw_0 = \tilde{w}_0$, and moreover u_0, $\tilde{v}_0$, $\tilde{w}_0$, p_0, η_0 are real. In this case it should already be taken, that the average p, values also depend on latitude, since they are bound up with the wind by geostrophical relations. The condition of statics should also be met. Thus

$$U = \frac{1}{\rho 1a} \frac{\partial \bar{p}}{\partial \Theta}, \quad \frac{\partial \bar{p}}{\partial z} = - g\bar{\rho}$$

If it is assumed, that at U there is a minor parameter, then the quantities $\bar{p}$ and $\bar{\rho}$ may be taken as expanded according to orders of this parameter; $\bar{p} = \bar{p}_o + \bar{p}_1$, $\bar{\rho} = \bar{\rho}_o + \rho_1$. The terms of zero order depend only on vertical coordinate z. We identify them with the standard CIRA atmosphere 1961, which we were using up to now. From the condition of geostrophicity we determine by integration from Θ value of the first order p_1 with accuracy up to random function from z

$$\bar{p}_1 = - 2a\omega \bar{\rho}_o \int_0^{\mu} \frac{\mu}{\sqrt{1 - \mu 2}} U_{\mu} d\mu + f(z), \quad (\mu = \cos \Theta).$$

To specialize this function we require, additionally, that the mean $\bar{p}$ value would coincide in latitude with the pressure in standard atmosphere, i.e. $\int_{-1}^{1-} p_1 d\mu = 0$. Then we get

$$\bar{p}_1 = a\omega\bar{\rho}_o \left[\int_{\mu}^{1} \frac{2\mu}{\sqrt{1 - \mu 2}} U \, d\mu - \int_{-1}^{1} \sqrt{\frac{1+\mu}{1-\mu}} \mu U \, d\mu \right].$$

Hence, it is already possible to find p_1, using the equation $\partial \bar{p}_1 / \partial z = g\bar{\rho}_1$. Thus, the wind, i.e., function U(z, Θ), could be set arbitrarily, whereas the other quantities, characterzing the main flow, should be determined unambiguously.

If we now use the same procedure of the theory of disturbances, as in the case of flat earth, we will get the following result:

C-4

$$\sigma_1 E = -\frac{s}{a}\int_0^{\pi}\int_0^{\infty} Ue\ dz\ d\Theta + \tfrac{1}{2}\int_0^{\pi}\int_0^{\infty}\Big\{ = \sigma_o \bar{p}_1\big(u_o^2 + \tilde{v}_o^2 + \tilde{\omega}_o^2\big) +$$

$$+ \bar{p}_o\left(\frac{U'_\theta\ \tilde{v}_o\ u_o}{a} + U'_z\tilde{\omega}_o u_o\right) + \left(\frac{lU'_z}{\beta}\right)\tilde{v}_{o\eta\,o} + 2lU\rho_o\tilde{v}_o + 2l\tilde{p}_1 u_o\tilde{v}_o -$$

$$-\left(\frac{g\bar{p}_1}{x\bar{p}o}\right)\tilde{\omega}_o p_o + \left[g\bar{p}o\frac{(\beta\rho)_1}{(x\bar{p}\beta\rho)_o}\right]\omega\tilde{o}\eta o -$$

$$-\left(\frac{\bar{p}_1}{po}\right)\tilde{\chi}opo\Big\}\ \sin\Theta\ dz\ d\Theta. \qquad (8.4)$$

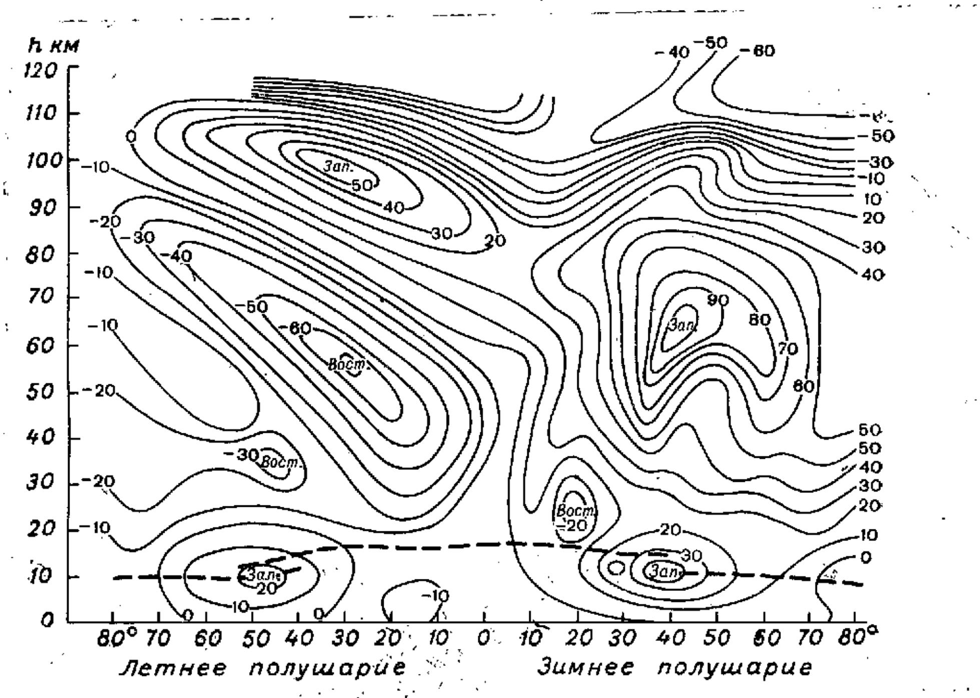

Fig.8.1 - Section of the middle zonal flow according to Murgatroyd (the continuous lines show velocity isopleths in m/sec., dotted lines - tropopause).

All the quantities here are real. This formula is the most common.

We have used this result for estimating on computer corrections for the frequency of natural oscillation, the nearest to semidiurnal, so as to see whether the estimate of wind would affect the conditions of resonance generation of forced and natural oscillations. As the model of wind we took Murgatryd's model (1957), shown in Figure 8.1, where the isotachs of moderate wind are shown in m/sec. The mean pressure was determined from the preset wind, as described above. Correction for frequency, obtained from formula (8.4), was found to be extremely negligible. It cannot be of any significance. Anyway, this correction obviously, depends to a great extent on the choice of the wind model. In any case it is clear, that by means of this correction it is impossible to explain the stable resonance of solar ties.

SUMMARY

The book deals with a study of the adiabatic oscillations of the rotating stratified atmosphere, the temperature of which depends upon the height, according to the so-called standard atmosphere. These oscillations can exist due to several different by their nature and, however, interacting factors - air elasticity connected with its compressibility, density stratification, and gyro-scopic rigidness due to rotation. Respectively, for every particular oscillation, i.e., at a definite value of frequency and wave numbers, as it appears, only one of these factors plays a determinative role; the rest introduce only small distortions. It enables to classify the oscillations rather distinctly, selecting those of acoustic, gravity and intertial-gyroscopic types.

Much attention has been paid to a study of the energetic structure of oscillations. Primarily, it appears of interest to find the relation of energy parts of different types: kinetic, connected with the horizontal components of velocity, the same with the vertical component, further elastic, connected with pressure oscillations, and thermobaric - with entropy oscillations. This energetic structure characterizes the type of oscillations in the above sense; it determines also the group velocity of waves.

The paper deals with space distribution of the energy density. The latter seems to be important in a study of wave quide properties of the atmosphere as related to short waves, and also in a study of thermal

barriers preventing the energy escape upward for very long waves.

One manages to study some non-stationary problems of dynamics of the atmosphere by way of the extension on the wave solutions. Namely, one treats the wave propagation from the instantaneous point disturbance like a strong explosion. The solution splits, naturally, to the acoustic and gravity parts. In accordance with large group velocities of acoustic waves, the first part of the solution disperses quickly. The second one disperses much slower, forming the oscillating "tail" of the wave. The asymptotic formulas for the solutions for large time have been found and compared with exact computations by an electronic computer.

One more physical problem treated in the book is a study of perturbations in some parts of the oscillation spectrum created by the mean zonal wind.

Very much attention has been paid to the development of the mathematical apparatus of the theory. Particularly, two chapters of the book have been devoted to the study of one of the two fundamental equations of the theory - Laplace tidal equation. The spectrum of the eigenvalues of this equation has been completely studied. The asymptotics for the limiting values of the parameters have been found. The mathematical properties of the second equation of the theory involving the characteristics of the vertical structure of the oscillations have been studied. The study of energy turns to be of interest not only for itself as the most important physical characteristics, but it gives the mathematical apparatus - the natural metrics for functionally-analytic investigation of the equations. In particular, one manages to clear up the question on completeness of the system of wave solutions (that is important to establish the possibility to solve the non-stationary problems by the Fourier method). The same metrics permits to solve the problem on the perturbations of the spectrum by the mean zonal wind.